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BALANCE OF PAYMENT EQUATIONS AND EXCHANGE RATE DETERMINATION

by

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I. Introduction

This paper is intended to elaborate and justify rigorously the statements made in Part III of "Modeling the International Influence on the U.S. Economy: A Multi-Country Approach," [2](the summary paper). The primary purpose is to justify the use of equilibrium conditions based on the balance-of-payments accounts in the determination of the exchange rates and other endogenous variables of the overall model developed in the summary paper. Here I shall show how such "balance-of-payments conditions or equations" can be constructed as a linear combination of the model's market clearing conditions and budget constraints. Then it will be proved that a balance-of-payments equation can be substituted for a judiciously chosen market clearing condition, while leaving the set of solutions for the system unchanged. Thus one can prove that the transformed model containing balance-of-payments conditions is <u>equivalent</u> to the original set of equations.

This equivalence will be proved below for two types of model. The second, which will be patterned as closely as possible after the model actually proposed in the summary paper, will contain some markets that can be in disequilibrium. This is a more difficult system to analyze, but nevertheless the same equivalence result can be proved. All of the theorems in this paper will be proved for period models, where observations and decisions are taken at discrete intervals of time. There is some question whether and under what conditions the results hold for continuous models; this potentially interesting question will not be pursued in this paper, since all available empirical data, and previous empirical work, fit the discrete-time framework.

II. System and Notation

The model and notation of this paper are patterned closely after those in Parts II,III and IV of the summary paper; however, since here the countries and markets must all be specified at once, it will be necessary to use somewhat more complicated subscripting. The world we propose to construct consists of 6 inter-related countries--the United States, Canada, Japan, the U.K., W. Germany and the rest of the world (ROW)--each of which contains five markets for the following commodities: a composite consumption and investment good (Q); labor (L); the monetary base (B); short term securities (STS); long term securities (LTS).[2] Associated with these markets are prices and interest rates; since commodities produced in different countries are imperfect substitutes in our system,there will be 29 separated prices, one less than the number of distinct commodities: 6 goods prices (Pj); 6 wage rates (Wj); 6 short-term interest rates (RSj); 6 long-term interest rates (RLj) and five independent bilateral exchange rates (Rj). "j" of course stands for the country in question; thus the goods price for the United

States is P1. Only the price of U.S. base money is not variable, being set at 1 and serving as the numeraire of the system. Table 1 shows the markets and associated prices.

The objective of the model-building is of course to build a system of equations that can be used to determine the 29 independent prices, the total amounts produced, and the allocation of each commodity to the various economic agents in the system.[3] Our system of equations, like all such systems, has the following pieces: first are the behavioral supply and demand functions for each agent; together they are combined into the supply and demand equations for each of the distinct markets whose prices will be determined endogenously (we could have as many as 29 endogenously determined prices; in our initial effort, all of the prices for ROW will be treated exogenously, leaving 24 endogenous prices). Besides the market supply and demand equations, we will also make use of country budget constraints, to be detailed below.

First consider the market supply and demand functions. Microeconomic theory tells us that the market demand for a particular commodity (at a given set of prices, incomes, etc.) is the sum of the demands of the individual agents in the system; similarly for market supply. The construction and properties of these agent and market functions is the province of Parts II and III of the summary paper and its companion papers.[4] For the purposes of this paper we need to know only that these market equations exist and are sufficient to determine an equilibrium for the system: i.e., provide solutions for the unknown prices, outputs and allocations.

Table 1

The Markets to be Modeled and Prices to be Solved for*

Markets \ Countries	1 U.S.	2 Canada	3 Japan	4 U.K.	5 W. Ger	6 ROW
Goods (Q)	P1	P2	P3	P4	P5	P6
Labor (L)	W1	W2	W3	W4	W5	W6
Monetary Base (B)	RS1	RS2	RS3	RS4	RS5	RS6
Long-Term Securities (LTS)	LTS1	LTS2	LTS3	LTS4	LTS5	LTS6
Short-Term Securities (STS)**	1	R2	R3	R4	R5	R6
Balance-of-Payments Equation (BOP)**						

*The price or interest rate symbol is associated in Table 1 with that market whose equation will be used to determine it; the only exception to that rule is noted in note **, below. In a truly simultaneous equilibrium system the prices could be determined, of course, in virtually any market.

**The balance-of-payments equations (BOP) will be substituted for 5 of the 6 STS market-clearing conditions; the 6th will be dropped because of Walras' law.

For each of our markets we shall aggregate the demand and supply functions by country. Thus, taking the goods market in the United States (country 1) as an example, we shall distinguish the aggregate demand for goods by all residents of countries 1 through 6, and likewise for the supplies. As far as notation is concerned, the demand for the U.S. good by agents of country 2 will be represented as follows: $Q1_2^d$. The symbol "Q1" means the composite good (Q) produced in country 1; the subscript denotes whose demand it is--in this case the demand of residents of country 2; the superscript "d" of course indicates it is a demand function or curve we are talking about. For supply functions we use the superscript "s", for excess demand functions (demand minus supply) the superscript "ed", and for excess supply functions the symbol "es." We shall use this notation both for the total demand supply, and excess demand in a given market and for the components of these market schedules: e.g. the excess demand of a given country in that market.

For each of the separate commodities (30 in number), we will use both a special commodity letter to designate the commodity type and a number to denote the country that produces it. As noted above the commodity letters are: Q for the consumption good; L for labor services; B for base money (currency and reserves); STS for short-term securities; and LTS for long-term securities. Thus, for example, the demand for U.S. base money by residents of the United States is: $B1_1^d$; the excess supply of country 2 commodities by residents of country 2 is: $Q2_2^{es} = Q2_2^s - Q2_2^d$.

For each market, price and quantities produced are determined either by (1) market clearing or (2) some special rules of disequilibrium pricing and production. In the next section, section III, we will assume that <u>all</u> markets clear; prices, production, etc. will be determined by the familiar process that for the market, supply equals demand or excess supply or demand equals zero. In section IV disequilibrium will be introduced into some of the markets. For our system, we have aggregate demand functions for each commodity by country, but only one ultimate supplier, who resides in the country of origin; thus, for markets that clear, the market clearing condition typically looks like the following (e.g. for the country 1 good):

$$\sum_{i=1}^{6} Q1_i^d - Q1_1^s = \sum_{i=1}^{6} Q1_i^{ed} = Q1^{ed} = 0$$

A. The Budget Constraints

We know from micro-economic theory that the demand and supply functions of a given agent are not completely independent. For each agent, the functions must be such as to satisfy his budget and/or wealth constraint. By the "budget" constraint we mean the well-known constraint on an agent's transactions during a given period of time. One way to express this is to say that, for a given period, at any given set of prices, an agent's demand and supply functions must be such that his sources of funds must be equal to his uses; sources of course can come from work, selling existing stocks of commodities and assets, and borrowing (issuing securities); typical uses are for consumption and the acquisition of assets.

The wealth or balance-sheet constraint refers to the identity between total assets and the sum of total liabilities and net worth that must hold at every point in time. Naturally these two constraints are not independent; the holding of the wealth constraint at two points in time implies that the budget (flow) constraint must hold over the period spanned by these two points. Another way of expressing the interdependence of these two constraints is to say that the value of wealth (net worth) at the end of the period must equal the value of wealth at the beginning of the period, plus savings and capital gains during the period.[5]

In Part II of the summary paper budget and balance-sheet constraints are presented for each actor in a typical country. To link that treatment to the present one, let us consider one of the constraints, e.g., the budget or flow constraint for the private sector of (say) the United States, introduced in section A.3 of Part II:

$$(1) \qquad R_P - E_P = \Delta CUR_P + \Delta DD_P + \Delta TD_P + \Delta FA_P + \Delta STS_P + \Delta LTS_P.$$

There are several notational differences between this paper and the summary paper. First, the constraints in the summary paper are expressed directly in value terms, whereas, below we will frequently separate prices and quantities. Second, a primary concern of the summary paper is to emphasize distinctions among the different agents; hence the use of the subscript "P" to denote private,and no subscript to denote the country of residence. In the present paper, we shall aggregate over all agents in a country, so agent subscripts are rarely necessary. No country indicators appear either for goods or actors in the summary paper because (1) the major part of that paper is concerned with describing the prototype sub-model for a single country, (2) attention is paid to only one agent in a given country at a time and (3) all foreign goods and assets are aggregated for convenience into the ΔFA_P term (the change in all foreign assets) and the $R_P - E_P$ term (the aggregate of receipts minus expenditures). For the present paper, on the few occasions where we will need to distinguish among agents in a given country we will use a double subscript, the first referring to the country and the second, a letter, referring to the agent; thus, for example, ΔSTS_{1P} will denote the change in short term securities held by the private sector in country one.

The budget constraint expressed in equation (1) most obviously holds ex post: the observed value of the sum of asset changes will always be equal to the surplus (or deficit) of receipts minus expenditures from the goods and labor market -- plus such receipts as interest earnings. Again, sources must equal uses of funds. However, the constraint in (1) must hold ex ante as well, at all possible sets of prices and interest rates. The construction and use of ex ante budget constraints is well known in models of consumer choice limited to flows of goods; however, the complexity increases when the holding of assets is introduced. Let us illustrate with one of the asset change terms in equation (1), the ex ante change in the holding of short-term securities, ΔSTS_p. Ex ante asset changes corresponding to ΔSTS_p equal the agent's excess demand: his demand minus the amount he can supply (in most cases this latter is just the stock the agent has accumulated in the previous period); thus $\Delta STS_p = STS_p^d - STS_p\ (t-1)$. In this section we will make the further assumption that equilibrium is achieved in every period; in this case the lagged <u>observed</u> stock, $STS_p(t-1)$, equals the ex ante demand at the prices reigning at time t-1: $STS_p^d\ (t-1)$. This assumption will be relaxed below when disequilibrium is allowed.

For a supplier of STS, possibly any or all of the government, banks private firms and individuals, one could just utilize the convention that negative holdings imply a net supply; however, to maintain the links with the summary paper, supply functions will be specified separately. A supplier of STS would have the following entry for ΔSTS_p (in value terms): $-(STS_p^s - STS_p^s(t-1))$. If, by chance, the actor or sector <u>both</u> supplied and demanded the same asset STS, we could combine the two sets of transactions into:

$$\begin{aligned} (2) \qquad \Delta STS &= STS^d - STS^d(t-1) - [STS^s - STS^s(t-1)] \\ &= STS^{ed} - STS^{ed}(t-1). \end{aligned}$$

Let us now link the other entries in the private sector constraint of the summary paper, equation (1) above, to the markets we have defined above. ΔSTS of equation (1) corresponds to $\Delta STSj_j$ in the nomenclature defined above--for country j, the change in the holdings of the domestically produced STS by domestic agents. As noted above, the entries like $\Delta STSj_j$ in this paper are aggregatedover all agents resident in a given country. Similarly, ΔLTS in equation (1) corresponds to $\Delta LTSj_j$.

The markets for demand and time deposits, ΔTD_p and ΔDD_p, are in some sense being ignored in this paper. As shown in the companion paper by Clark and Kwack,[6] these markets are really just part of the larger market for base money-- which we will use proximately to determine the short-term interest rate (RS). For our purposes it will do no hard to combine demand and time deposits with base money; hence we shall substitute ΔB for $\Delta CUR + \Delta DD + \Delta TD$ in the summary paper.

The final term on the right-hand side of the budget constraint (1) is ΔFA_p, the change in actual or, alternatively, desired holdings of <u>all</u> foreign assets. In terms of the notation of this paper, the aggregate for citizens in the United States (country 1) of ΔFA equals:

$$(3) \quad \sum_{j=2}^{6} Rj\,\{Bj_1^{ed}-Bj_1^{ed}(t-1)\}+\sum_{j=2}^{6} Rj\,\{STSj_1^{ed}-STSj_1^{ed}(t-1)\}+\sum_{j=2}^{6} Rj\,\{LTSj_1^{ed}-LTSj_1^{ed}(t-1)\}.$$

Thus this term is the sum of a potential of 15 asset holdings: 3 types of assets for each of 5 foreign countries. Note that to change the holdings into dollar values, one multiplies by the current exchange rate, Rj, between the dollar and the relevant foreign currency.

Finally, let us decompose the net receipts term on the left-hand side of (1). This combines all activity in the goods and labor market, plus transfers and interest earnings from holdings of assets. For residents in the United States net receipts are primarily from labor services ($W1 \cdot L1_1^s$), the sale of current production ($P1 \cdot Q1_1^s$), and interest income--a typical item of which is $RS1 \cdot STS1_1^d$. Expenditures are for goods of all kinds ($P1 \cdot Q1_1^d + \sum_{j=2}^{6} Rj \cdot Pj \cdot Qj_1^d$) for purposes of consumption, investment, labor services demanded, the payment of interest (e.g., $RS1 \cdot STS_1^s$), and taxes.[7] (When we aggregate over all agents in a country we will assume that taxes and transfers net out.)[8] This finishes our linking of a typical agent's budget constraint in the summary paper with our market notation.

B. Deriving Country Budget Constraints

Now we wish to add the budget constraints for all agents resident in a given country; the result will be the country budget constraint we will work with below. After their derivation, these country budget constraints will be linked to the country excess supply equations appearing in each market.

The budget constraint for a given country is just the sum of the budget constraints for each agent resident in the country. However, there can be complications in doing that summation. Asset changes are particularly tricky. Consider the summation over all agents in country 1 for the change in a typical asset such as ΔSTS expressed above in equation (2). As noted above, in this case we will resort to double subscripting: the first, referring to country 1; the second, "i", to the agents resident in that country.

$$\sum_i \Delta STS1_{1i} = \Delta STS1_1 = STS1_1^{ed} - STS1_1^{ed}(t-1)$$

But that is not all; for later use in deriving balance-of-payments equations, it is important to note that the last term in ΔSTS1 equals:[9]

$$(4) \qquad -STS1_1^{ed}(t-1) \equiv + \sum_{j=2}^{6} STS1_j^d(t-1).$$

The lagged excess supply of U.S. residents for a home-produced asset necessarily equals the lagged foreign demand for that asset. To elaborate a little more, this must be true for the past period, whereas it need not be true ex ante for the present period. It is true for past periods either (1) because we are assuming that actual past transactions represent an equilibrium; thus the sum of demands equals supply; or (2) because for lagged transactions, we can operate solely in terms of observed quantities; thus I think we can drop the assumption that $STS1_1(t-1)$ observed last period was on the excess demand function for that period; it is always true ex post that observed home demand minus supply equals observed foreign demand. This ability to allow past transactions to be off demand and supply curves will be useful when we get to the modeling of disequilibrium situations in section IV.

Thus, to summarize our representation of a country's (ex ante) change in the holding of a domestically-produced asset, one gets the following (using STS1 as a concrete example):

$$(5) \qquad \Delta STS1_1 = STS1_1^{ed} - STS1_1^{ed}(t-1) = STS1_1^{ed} + \sum_{j=2}^{6} STS1_j^d(t-1)$$

For a country's holding of an asset produced abroad we have the following (using the U.S. holdings of the short-term asset produced in country 2 as an example):

$$(6) \qquad \Delta STS2_1 = STS2_1^d - STS2_1^d(t-1) = STS2_1^d - \left(\sum_{j=2}^{6} STS2_j^d(t-1) - STS2_2^s(t-1)\right)$$

As long as we remember that only j produces STSj, we could use the same expression for $\Delta STS1_1$ and $\Delta STS2_1$, replacing every "d" with "ed" [except for the last term in (6) which equals $\sum_{j=2}^{6} (STS2_j^{ed}(t-1)]$. The expression in (6) is in units of foreign currency; to enter the budget constraint of the United States it must be multiplied by the exchange rate, R2.

For the demand for the consumption-investment good we have no problem. The term in the country's budget constraint for its own good is: $P1(Q1_1^s - Q1_1^d)$; for another country's good, since residents of country 1 do not produce such goods, the term is merely: $Rj \cdot Pj \cdot Qj_1^d$. Summing, net receipts from all goods are represented as:

$$(7) \qquad P1(Q1_1^s - Q1_1^d) - \sum_{j=2}^{6} Rj \cdot Pj \cdot Qj_1^d \; .$$

A similar set of terms appears for net receipts from all labor services (sold and purchased):

$$(8) \qquad W1 \cdot (L1_1^s - L1_1^d) - \sum_{j=2}^{6} Rj \cdot Wj \cdot Lj_1^d \; .$$

The last term in (8) allows for expenditures for foreign labor--particularly important for countries like Germany. Note that the first term in (7) and (8) <u>in equilibrium</u> will equal exports--foreigner's demand for the home good. However, where the prices in the functions are not equilibrium prices, the excess supply (e.g., $Q1_1^s - Q1_1^d$) <u>will not</u> be equal to the foreign demand for the good.

The final set of terms in the country's receipts entry deals with net interest. For a foreign asset, of which there are 10 that bear interest, a typical term is:

(9) $Rj \cdot RSj \cdot STSj_1^d$

We assume that interest is paid on current holdings; an alternative that would not change the theorems that are deduced below would be to assume that interest is paid on lagged holdings at the lagged rate of interest--i.e., that you don't get your interest until the next period.

For the home-supplied assets, domestic holders get receipts, but the issuer must pay out a flow equal to the interest rate times the total stock. Thus <u>net</u> interest receipts for the country on this asset equal:

(10) $RS1 \cdot (STS1_1^d - STS1_1^s)$.

Naturally, <u>in equilibrium</u>, the term in parentheses equals foreign holdings of the asset.

Putting it all together, the budget constraint for country 1 becomes (in all its glory!):[10]

(11) $$P1(Q1_1^s - Q1_1^d) - \sum_{j=2}^{6} Rj \cdot Pj \cdot Qj_1^d$$ (goods)

$$+ W1(L1_1^s - L1_1^d) - \sum_{j=2}^{6} Rj \cdot Wj \cdot Qj_1^d$$ (labor)

$$+ RS1(STS1_1^d - STS1_1^s) + \sum_{j=2}^{6} Rj \cdot RSj \cdot STSj_1^d$$ (net interest receipts on short-term securities)

$$+ RL1(LTS1_1^d - LTS1_1^s) + \sum_{j=2}^{6} Rj \cdot RLj \cdot LTSj_1^d$$ (net interest receipts on long-term securities)

$$- [(STS1_1^d - STS1_1^s) + \sum_{j=2}^{6} STS1_j^d(t-1)] - \sum_{j=2}^{6} Rj[STSj_1^d - STSj_1^d(t-1)]$$ (minus net expenditures on short-term securities)

$$- [(LTS1_1^d - LTS1_1^s) + \sum_{j=2}^{6} LTS1_j^d(t-1)] - \sum_{j=2}^{6} Rj \cdot [LTSj_1^d - LTSj_1^d(t-1)]$$ (minus net expenditures on long-term securities of all kinds)

$$- [(B1_1^d - B1_1^s) + \sum_{j=2}^{6} B1_1^d(t-1)] - \sum_{j=2}^{6} Rj \cdot [Bj_1^d - Bj_1^d(t-1)] \equiv 0.$$ (minus net expenditures on monetary bases of all kinds)

III. Solving the Equilibrium System With and Without Balance of Payments Equations

We now have the pieces to solve our system of 6 countries with five distinct goods for each country. A solution consists of the determination of 29 prices, 30 levels of total production, and the allocation of this production among the agents in each country. In this paper, since we have aggregated over agents in a given country, we will determine the allocation by country only, rather than by each actor in a given country.

What do we have to work with to solve for these prices and quantities? First of course we have the demand and supply functions for each of the 30 commodities by country. Second, there are the budget constraints for each country, so laboriously constructed in the last section; thus, there are 6 constraints--identities which hold at all prices and outputs--of the form of equation (11). The budget constraint for country "i" is the same as (11) except that subscript "i" is substituted for subscript 1 and vice versa; also, one must pay attention to interchange "i"'s and 1's in the names of the commodities.

Third, we have the market clearing conditions for each market. Since we will assume for this section that each market price will be determined by the equating of supply to demand, we have 30 equations of the following form (using the market for the U.S. good as an example):

$$(12) \qquad Q1_1^s - \sum_{j=1}^{6} Q1_j^d = 0 = Q1_1^{es} - \sum_{j=2}^{6} Q1_j^d .$$

If the thirty equations of the form of equation (12) were independent, we could determine 30 unknown prices (as well as the quantities that are a function of these prices). However, as is well known, because of the budget constraints these 30 equations should be dependent; only 29 of them are independent--and these should be sufficient to determine the 29 prices in Table 1.

To verify that our system has been constructed correctly, let us prove this dependence. Sum the 6 budget constraints of the form of equation (11); note that they must all be expressed in one currency--for convenience, dollars, as is the U.S. constraint in (11). What happens for each market as we sum? Consider the market for Q1. From the U.S. constraint, since the U.S. produces the good, we get $P1(Q1_1^s - Q1_1^d)$. From the constraints for other, non-producing, countries we get terms of the following kind: $P1 \cdot Q1_j^d$--one for each country j; thus if there is market clearing in this market (from equations (12)), the sum of these terms equals zero. This kind of argument holds for all goods markets and labor markets.

What about interest receipts? The market clearing condition for asset markets is that the sum of stock demands equals the supply of the stock--just as in (12). If we add the receipts term for interest on short-term U.S. securities, the term beginning with RS1, we get: $RS1(STS1_1^d - STS1_1^s - \sum_{j=2}^{6} STS1_j^d)$. Thus if the market clears this term equals zero also. Similarly for net receipts from all other securities.

Finally what about the sum of the asset market terms? Consider the terms for STS1. Adding over all countries we get:

$$- [(STS1_1^d - STS1_1^s) + \sum_{j=2}^{6} STS1_j^d(t-1)]$$

$$- \sum_{j=2}^{6} [STS1_j^d - STS1_j^d(t-1)]$$

This simplifies to $STS1_1^s - STS1_1^d - \sum_{j=2}^{6} STS1_j^d$. If the market clears the sum of these terms is also zero. This leads to the typical proof of the dependence of the market clearing conditions. The sum of the country budget constraints is identically equal to zero; grouping the terms by market, the above calculations show that if n-1 markets clear (in our case 29), the sum of the budget constraints reduces to either:

$$P1\, Q1_1^s - \sum_{j=1}^{6} Q1_j^d \equiv 0$$

$$\text{or} \quad (1-RS1)\left\{STS1_1^s - \sum_{j=1}^{6} STS1_j^d\right\} \equiv 0$$

depending on whether the last market is a good or asset market. In both cases the budget constraint says that this market must be in equilibrium too; hence there are at most 29 independent market-clearing conditions.

Before going to the construction and substitution of balance-of-payments equations, let us recall exactly how one proceeds in solving this sort of standard general equilibrium system. First one picks any 29 of the 30 market-clearing conditions and solves for the 29 prices; given the 29 independent prices one then has all that is required to calculate the supplies and demands for each country and agent. For the calculation of the demands and supplies in the market that was dropped in the calculation of the equilibrium prices, one can use either the market-clearing condition for that market or the budget constraints; the results must be identical.

A. Substituting Balance-of-Payments Equations for Market-Clearing Conditions

We will assume that the above system of 29 independent market-clearing conditions and the six country budget constraints is sufficient to determine all the prices and quantities. This we will call the basic system. The reader should note that the balance of payments appears nowhere in this system; we have worked directly with the 30 individual markets and all was well. However, in practice, as discussed in the summary paper (part II), there are a number of empirical reasons why one cannot estimate the basic system. We thus search for an estimable system that is equivalent to the basic one--by equivalent is meant a system of equations that always yields the same set of prices and outputs as the basic one.

One such equivalent system that I think can be estimated empirically involves the use of balance-of-payments equations <u>in place of</u> certain market-clearing conditions. In particular, in this section, equations that are ex ante balance-of-payments equations will be substituted for the market-clearing conditions in every short-term securities market. More precisely, we will drop one such market because of Walras' law and substitute balance-of-payments conditions for the five remaining market-clearing conditions.

1. The Derivation of Balance-of-Payments Equations

Our definition of the balance of payments will be the usual one; the balance of payments equals the sum of exports minus imports of goods and services (including interest receipts) plus net capital flows. However, it should be remembered that we are working exclusively with demand functions and supply functions, so that the equation will be an ex ante one.

Let us start from the budget constraint for the country, (11). In one sense constraint (11) already is a representation of the balance of payments: for a given set of prices, if the country could allocate the excess supply of its home goods and securities to foreigners and achieve its desired flows of foreign goods and holdings of foreign securities, budget constraint (11) would equal the balance of payments of country 1--and since it always equals zero, the balance of payments would equal zero.

However, an identity cannot be used to substitute for an equation --which is our purpose. Thus constraint (11) must be transformed into an equation. To do this we shall adopt the convention that for a country's domestically produced goods and assets, we will measure the ex ante balance of payments flow from the foreign demand side; thus for the first term in (11) we substitute: $\sum_{j=2}^{6} P1 \cdot Q1_j^d$. This is equivalent to subtracting the functions in the market clearing condition (12) from the first term in (11). That is to say:

$$P1(Q1_1^s - Q1_1^d) - [P1\ (Q1_1^s - Q1_1^d) - P1 \sum_{j=2}^{6} Q1_j^d] = \sum_{j=2}^{6} Q1_j^d.$$

A similar substitution or subtraction is made for the first term in each line of (11) --all terms pertaining to the goods and services produced by country 1. Since everything in (11) pertaining to foreign commodities is already measured from the demand side, the general convention leads to the result that all items in a transformed budget constraint will be measured from the demand side.

Consider the above substitution for the first term in (11). For some sets of prices the market will not clear: the sum of foreign demands will not equal the U.S. excess supply of Q1. In this case the term subtracted from $P1(Q1_1^s - Q1_1^d)$ will not equal zero, and the transformed version of (11) will likewise no longer equal zero. Thus the transformed version of (11) is no longer an identity.

Further it is easy to show that this balance-of-payments equation equals zero when prices are equilibrium prices. For in this case, the market clearing condition assures that the terms subtracted from identity (11) are all equal to zero. Thus the sum of the identity, which always equals zero, minus a series of terms each equal to zero, equals zero.

Note that this seems to be a very natural way to construct a balance-of-payments condition; it is entirely from the demand side --and this corresponds to the way we usually construct equations for the flows in the balance of payments, both theoretically and empirically. For example, the demand for U.S. exports is usually constructed theoretically as the sum of foreign countries' demands for U.S. goods; this is the procedure that will be followed for the empirical model.

One final caveat. In equilibrium it doesn't matter from what side one constructs a balance-of-payments equation--because supply equals demand. However, out of equilibrium, balance-of-payments equations constructed different ways will give different results. For a disequilibrium model this may be a significant observation, for actual transactions out of equilibrium may be dominated by conditions of excess supply in some cases and excess demand in others. However, to reiterate, where equilibrium reigns, as long as we avoid identities, it doesn't matter from what side we measure a balance-of-payments flow.

B. The Proof of Equivalence

Consider the following as an alternative to the basic system: substitute 5 balance-of-payments equations for 5 market-clearing conditions in the short-term securities markets; as assumed previously, drop the 6th short-term securities market because of Walras' Law, This new system differs from the basic one only in that 5 BOP equations are added and five short-term securities market-clearing conditions are dropped. To derive equilibrium prices all the 29 equations in the alternative model are set equal to zero.

The following theorem can be proved:[11]

> All the solutions (prices and quantities) of the new system are solutions to the basic system; and all the solutions of the basic system are solutions to the new system.

Mathematically the two systems are equivalent. Further the result is quite a bit stronger than the one outlined above; in fact one can substitute the BOP equations for <u>any</u> of the markets whose clearing conditions are embedded in them.

The proof goes as follows. We note that each BOP equation is constructed as the sum of market-clearing conditions and one budget identity from the basic model. Thus as long as a BOP equation contains the market-clearing condition that it substitutes for, the new system contains all the equations of the old system--and no more new ones. This is the key to our theorem.

Let S be any set of prices for the basic model--i.e., for this set of prices all the equations of the basic model are equal to zero. Then S must satisfy the new model: it certainly satisfies the market-clearing conditions that appear in both models--all equations but the new BOP conditions. Does S satisfy the balance-of-payments conditions? Since each BOP condition is just the sum of an identity and market-clearing equations from the old model, S must satisfy the BOP condition also. Thus the second part of the theorem is proved.

The first part is similar. Let S' be a set of prices that solves the new model. We must show that S' solves the basic model too. It is immediately true for all equations common to both models that S' is a solution. The only question is whether S' solves the set of equations that appears directly in the basic model, but not in the new one--the short-term securities market-clearing conditions for the five countries. We can see that it does by looking at the BOP equations; by assumption S' satisfies each BOP equation; in each is embedded a short-term securities condition. But if S' satisfies the BOP condition and every other equation embedded in the BOP condition, then S' must also set equal to zero the short-term securities market-clearing condition (this is true because only one short-term securities market-clearing condition appears in each BOP equation).[12] Thus S' satisfies the old model. And the theorem is proved.

C. Some Further Comments

An interesting point is that, as we defined the BOP equations for the six countries, the sum of the six are linearly dependent: they in fact add up to zero, no matter what the reigning prices and interest rates. This can be seen by taking a typical market, say for the U.S. good, Q1; by defining the balance of payments from the demand side, we have already determined that the entry in the U.S. BOP equation for Q1 will be the sum of all foreign demands for Q1. On the other hand, for the Q1 market, one finds in each other BOP equation that country's demand for Q1--entered with a minus sign, since it is an import. Thus, adding all the entries in this market, they sum identically to zero; a similar result holds for all other markets. The implication, of course, is that the six equations cannot be used together; but above we used only five, because we needed only five. The omitted country's ex ante balance of payments will be equal to the negative of the sum of the other five countries.'

This result depends on the way we have represented the ex ante balance of payments; however, one need not follow the rule that I postulated above; in particular one need not substitute in every case for excess supply functions in domestic markets. Although the following point needs further investigation, it seems that the above summing of the BOP equations to zero depends on the particular way the substitutions are done.

IV. Solving the System When Some Markets are in Disequilibrium

In the previous section, the condition that all markets cleared--that supply was made equal to demand via price movements--played a very important part. It was an integral part of the proof that in equilibrium the balance-of-payments conditions equaled zero.

Unfortunately, however, the model presented in the summary paper does not have all markets clearing, every period. The major question for this section is to determine how disequilibrium affects the procedure for solving the model.

First, let us discuss the particular types of disequilibrium introduced into the proposed model. Elements of disequilibrium appear in both the labor and goods markets. Both are characterized by prices that do not move to clear the market within the space of a quarter--although they do move in response to excess demand, so as to clear the market eventually. As in all complete disequilibrium systems, once price does not clear the market, additional equations or assumptions must be introduced to determine how the discrepancy between demand and supply at the ruling price is resolved, and how allocations of the commodity in question are determined. The assumptions that are made about the two disequilibrium markets in the summary paper are as follows. For the labor market, we will adopt the usual assumption that labor is almost always in excess supply; in such cases the demand for labor rules and determines the quantity of labor services exchanged. In the goods market, we will assume that demand at the prevailing price is always satisfied--by a combination of the adjustment of labor demanded and changes in inventories. When an agent is forced off his ex ante function, we will say he is on his "effective" demand or supply function.

Whenever one ex ante demand or supply function is violated, the disequilibrium must spread to at least one other market; this is because the budget constraint says that there is a dependence among both an agent's ex ante excess demand functions and his ex post transactions, which in disequilibrium correspond to a combination of ex ante and effective functions. For laborers who are forced off their supply curves, we have followed the path of Clower [(5)], in postulating that laborers wait for actual income before determining their demands for commodities; thus all of the demand functions of workers are functions of observed income. The disequilibrium in the labor market is in this way spread over all markets. For firms, we assume that aggregate demand is almost always met. Labor and inventories adjust in some optimal way to make supply equal to demand. Other transactions of the firm will be affected also; we will attempt to incorporate these disequilibrium effects in our empirical work.

Let us end this introduction to the disequilibrium system by going over exactly how the available equations will be used to solve for the "disequilibrium" prices, outputs and allocations. First, the prices for the markets in disequilibrium are fixed for the period in question; the level for the present period is determined by a special price change equation (for the labor market) or a mark-up equation (for the goods market). The actual quantities produced or traded in this market are determined by further special disequilibrium equations for effective demands and supplies. Given the effective demand and supply function in other markets (different, possibly, from ex ante functions because of spillover effects), the prices in the other markets could be determined by the usual process of equating demand to supply. Since Walras' Law continues to hold there would be 17 such market-clearing conditions, instead of the previous 29. Given the solutions for the temporary or disequilibrium prices, the effective and, where appropriate, ex ante demand and supply equations are used to determine the quantities produced and traded. Finally, the budget constraint or, alternatively, the demand and supply equations for the redundant market are used to determine the demands and supplies in the market that has been dropped.

A. Balance-of-Payments Equations in Disequilibrium Systems

How does all this affect our use of balance-of-payments equations and the procedure for solving for exchange rates? There seems to be two different answers, depending on whether empirically one has only ex ante demand or supply functions to work with or, alternatively, effective demand and supply functions, as well as ex ante ones --the former reflecting the disequilibrium elements in the model. We shall cover the two answers in turn.

It is clear from the above discussion that we are following the second course in the empirical model--our empirical equations for the labor market, goods market, and those markets affected by the "spillover" of disequilibrium will be formulated to reflect the disequilibrium elements. Thus, when we formulate the budget constraint of a given agent and country it will contain disequilibrium effective demand and supply functions for certain commodities--as well as excess demand or supply functions for those markets that clear. For any given set of prices, etc., the budget constraint says that for an agent and a country, the functions in the budget identity sum to zero.

Suppose we proceed as in the last section to try to use a country budget constraint to construct a balance-of-payments equation. As was done before let us adopt the rule of substituting for all domestic markets in a given country's budget constraint. For markets that clear, we have no problem adding the market-clearing conditions to the budget constraint, and thereby replacing these excess supply elements by the sum of foreign demand functions--just as was done in the previous section.[13] This leaves, for our model, those domestic markets where the substitution of the last section cannot be made: the labor and domestic good market. However, given the proper effective demand and supply equations, measuring the disequilibrium effects in these markets, we still have a set of equations for each labor and goods market that serve to allocate the commodity in question. In particular, even though according to our ex ante demand and supply functions market clearing will not occur, the disequilibrium effective demand and supply functions will allocate goods and labor so that total effective demand for goods and labor equals total effective supply.

Using the effective demand and supply functions we can proceed very much as before in constructing the country's balance-of-payments equation. Thus, for the domestic goods market, for example, total foreign effective demand can be substituted for domestic effective supply. We then have a new balance of payments function for a given country: it is the same as the previous ones except that, where appropriate, effective demand and supply functions are substituted for ex ante ones. Further, one can show that when the system is in short-term or temporary equilibrium the balance of payments equals zero. The proof goes the same as in the previous section for markets in equilibrium. For those markets in disequilibrium--the labor and goods market--the effective demand and supply functions add up to zero. Hence the balance of payments equals zero when the system is in temporary equilibrium.

1. Using Ex Ante Conditions to Construct Balance-of-Payments Equations in Disequilibrium Systems

Above we have shown how "effective" demand and supply equations in situations of disequilibrium can be used in place of ex ante ones, both to construct valid balance-of-payments equations and to solve for the "temporary" equilibrium prices and quantities of the system. Theoretically this may be fine, but empirically we may have great difficulties estimating certain effective functions; a few examples of the difficulties encountered may be seen in recent work with trade equations and the domestic housing market. The latter has in work such as Fair and Jaffee [6] led to very complex estimation procedures to jointly estimate different equations for the cases where the participants are on their ex ante functions and where, because of disequilibrium, they are off them. In the area of international trade, the problems of estimating export and import equations in the face of capacity constraints and rationing, as studied by Henry [9], are problems of estimating the effects of disequilibrium. Given these difficulties, it may be the case that empirically the researcher will find it easier to estimate the ex ante functions than the effective ones.[14]

If the model builder is supplied only with ex ante functions, in some cases these can still be used to get the disequilibrium prices that solve the system.[15] Consider the simple case where only the goods market does not clear and where we know only the ex ante demand and supply functions for the good, and the equation for price changes; this latter is a simple function of the discrepancy between excess demand and supply:

$$(13) \qquad P1(t+1) - P1(t) = b(Q1_1^d + \sum_j Q1_j^d - Q1_1^s),$$

where "b" is a constant and the rest of the notation is as before. By dividing through by b (more generally, taking the inverse function), the excess demand in the market can be expressed as a function of the price change. In particular, domestic excess demand equals:

$$(14) \qquad Q1_1^d - Q1_1^s = 1/b[P1(t+1) - P1(t)] - \sum_j Q1_j^d .$$

Suppose now, that in trying to build up a balance of payments condition, we have proceeded as in previous sections and for all domestic markets but the goods market have substituted the total effective foreign demand for domestic excess supply.[16] Since these markets clear, as we have proved above, this equation will equal zero when the system is in temporary equilibrium. This equation could be used as a balance of payments equation and substituted for some other equation in the system; moreover, the domestic goods market is expressed in terms of ex ante functions--although it now is ex ante U.S. excess supply that appears here from the budget constraint.

What if one wanted to go a step further and, as above, substitute foreign excess demand in the domestic goods market for ex ante domestic excess supply?

Although the market no longer clears, we have in equation (13) a function which relates the total excess demand or supply to price changes in the market. In particular, using (13) or (14), for the domestic excess supply in our balance of payments equation we can substitute $\sum_j Q1_j^d - 1/b[P1(t+1) - P(t)]$. However, the equation for the (ex ante) balance of payments developed in the market clearing case, and also used traditionally, contains only $\sum_j Q1_j^d$, the first part of the previous substitution. Thus for the disequilibrium system, the balance of payments defined in terms of ex ante functions will not equal zero, but rather, $1/b[P1(t+1) - P1(t)]$. A balance of payments expression defined in this way could be substituted for some market clearing equation, as an alternative to the balance of payments equation defined in terms of effective demand functions.

B. Substituting Balance of Payments Conditions for Market-Clearing Conditions in Solving the Disequilibrium System

Suppose we want to substitute BOP equations for some market clearing condition in solving the disequilibrium system--as done above for the equilibrium model. Can we show that the solutions are equivalent? The answer seems to be yes, although there is one complication. Here we will consider only cases where BOP equations are constructed from effective demand and supply functions.

To the extent that the BOP equations are linear combinations of the market-clearing conditions of the basic model, the proof in section III for equilibrium systems goes through unchanged. However, there are two equations contained in the BOP equation developed above that are not in the basic set of market clearing conditions, the ones for the labor and goods market; these are the sum of the effective demand and supply functions for the markets that do not clear. In this system, recall that, for the purposes of price determination, the system is divided into two parts: the disequilibrium part and the set of markets for which there is market clearing. The BOP equations are substituted into this latter subset of equations, and the effective demand and supply equations in the disequilibrium markets--labor and goods--are not a part of this subset. However, these effective demand and supply equations are contained in the BOP equation. Hence to show the equivalence of the two systems, we must show that these added equations do not change the set of solutions.

In fact I think we can show this, but I am somewhat uncertain about the result. If we can show that the equation for "market clearing" in the disequilibrium markets is satisfied for all sets of prices in the equilibrium markets, or at least all sets that satisfy the basic system, then we can assert that the presence of this equation does not change the set of solutions. The way we define the effective demand and supply equations in the disequilibrium markets seems to imply this. For example, in the labor market, effective supply is defined to always equal the market demand for labor--the pool of unemployed assures this. For the goods market, we assume that inventories and unemployed labor are sufficient to allow the firm to satisfy any level of demand. Thus it seems permissible to assert that the equality of effective demand and supply in both these disequilibrium markets holds for <u>all</u> sets of prices. And thus equations from these markets do not constrain the set of possible solutions to the subset of market clearing equations.

From this point the proof of the equivalence of the two systems proceeds as in section III, above.

Footnotes

[1]It is a pleasure to acknowledge that this paper builds on work of Dale Henderson, who has rigorously derived a correspondence between a balance of payments condition and the market clearing conditions of certain of his models. The work here attempts to generalize that result and to extend the treatment to disequilibrium systems. This latter has been done jointly with Dale Henderson.

As in all the other papers of this series (see footnote 3, below, for titles), the members of the Quantitative Studies Section--Richard Berner, Peter Clark, Howard Howe and Sung Kwack--provided invaluable contributions while these ideas were being developed. Finally, I should like to thank Jeffrey Shafer, both for his helpful comments on all aspects of the work of the Quantitative Studies Section and for the stimulation provided by his own model of an open economy (reference [10]). Naturally, the views expressed herein are mine alone and do not necessarily represent the views of the Federal Reserve System.

[2]The specification for the rest of the world (ROW) proposed in the summary paper makes most variables in ROW exogenous. This step was taken because of problems of data availability; it can be modified at a later stage of our research.

[3]The summary paper outlines a model containing equations for four agents in each country: the government (or public sector), the central bank, commercial banks, and the private non-bank sector (households and firms).

[4]See the papers by Berner [1], Peter Clark and Sung Kwack [3], and Howard Howe [8], listed in the references.

[5]It might be noted at this point that for the asset demands we are assuming the agent "balances at the end of the period"--i.e., his choice of assets for period t is affected by his savings <u>during that period</u>. See Foley [7] for a discussion of the alternatives. On the basis of work done by Buiter [3] and Dale Henderson, we feel that "balancing at the end of the period" is preferable to balancing at the beginning of the period--despite what Foley says. In particular this alternative at least allows for the possibility that current exchange rates can affect trade flows.

[6]The companion papers, in addition to this present one, are entries [1], [4], [8], and [11] in the References.

[7]For paper assets like STS1, the change in the value held appearing in the budget or wealth constraint was written in terms of the first difference of stocks. Part of the flow of expenditures for goods is of course for real investment; this part is equal to:
$P1 \cdot K1(t) - P1 \cdot K1(t-1) + dP1 \cdot K1(t-1)$, where K is the real stock of capital (in units of the consumption good) and d is the depreciation rate.

[8] To introduce taxes from or transfers to foreigners should add no complications; the number of markets and prices is not altered; some equations are altered.

[9] It would seem that one need not assume that the lagged holdings were on the ex ante excess demand schedule: i.e., one can just as well work with observed, ex post, holdings at time t-1. If one does this the essential fact of equation (4) still holds--that the difference between (observed) domestic demand and supply equals foreign demand. Thus it does not seem necessary to assume that agents were on their excess demand curves in t-1 to derive their ex ante desired changes in asset holdings.

[10] See Table 1 above for a summary of the notation.

[11] The fact that the two systems must and do have the same number of independent equations is implicit in the proof of the theorem. See footnote 12, below.

[12] The BOP condition equals an identity, $I(P) \equiv 0$, plus a set of market clearing condition $f_i(P)$: $BOP(P) \equiv I(P) + \sum_{i=1}^{5} f_i(P)$, where P is any vector of the prices (and interest rates) of the system; "i" ranges over all the 5 domestic markets of the country in question.

The first part of the proof indicates that $BOP(S') = 0$ and for all but one of the $f_i(P)$, $f_i(S') = 0$. Hence for the one remaining $f_i(S')$, the short-term securities market, say $f_1(S')$, we have $f_1(S') = BOP(S') - I(S') - \sum_{i=2}^{5} f_i(S')$; therefore $f_1(S') = 0 - 0 - 0 = 0$.

This proof can break down in certain cases. Let us illustrate with an example: suppose that, instead of substituting 5 BOP equations (or generally one less than the number of countries or regions), we substitute <u>all six</u> BOP equations into the original model. We have assumed that the 5 BOP equations used previously replaced 5 short-term bond market-clearing equations; the 6th such equation was dropped because of Walras' Law. Thus, if a 6th BOP equation were to be used, it would have to substitute for a market <u>other than the short term bond market</u>. Let us say it will be substituted for the domestic money market of the 6th country. Let us denote market 1 as the domestic bond market and market 2 as the domestic money market. Thus $BOP6(P) = I6(P) + f_1(P) + f_2(P) + \sum_{i=3}^{5} f_i(P)$. Now $f_1(P)$ has been dropped from both the basic and transformed model because of Walras' law. Since we are substituting $BOP6(P)$ for $f_2(P)$ -- the market clearing condition for the domestic money market -- we must show that solution S' to BOP 6 (P) and $f_i(P)$, i=3, 5, must also solve f_2 (P). Note the difference: <u>two</u>, rather <u>than one</u> equation that are embodied in BOP6(P) no longer appear elsewhere in the transformed system. In this case we can not prove that S' is solution for $f_2(\ )$.

This can be demonstrated as follows. Because of Walras' Law, we know that $f_1(P)$ equals minus one times the sum of all other market clearing conditions. $f_1(P) = - \sum_{i=2}^{29} f_i(P)$.

Included in this sum is $f_2(P)$, the condition for the money market in country 6. Thus, substituting for $f_1(P)$ in BOP 6 (P), we see that $f_2(P)$ cancels out. Thus BOP 6 (P) is not really a function of the domestic money market in this case; hence there are price vectors P' such that P' does not solve the money market, but does solve BOP 6 (P) and all other market clearing conditions of the original model. Remember, since $f_1(P)$ was dropped, it was not used in the original model to solve for prices. Another way to state how the proof breaks down is that if we express:

$$f_2(P) = \text{BOP } 6\ (P) - I6(P) - f_1(P) - \sum_{i=3}^{5} f_i(P),$$

then we can have a P' such that BOP6(P')= 0, and $f_i(P') = 0$ for all i in the original basic model (recall that $f_1(\ )$ was not), and $f_2(P') \neq 0$. For any such P' the left hand side above will not equal zero (unlike in the independence case), for it contains $+ f_2(P') \neq 0$ in $- f_1(P)$.

This argument demonstrate that we cannot substitute a BOP equation for a market-clearing equation in that country where a market is dropped because of Walras' Law.

[13]Some of the component functions may be changed by "spillover" effects.

[14]The use of dummy variables for periods of disequilibrium is one way of estimating only an ex ante function; another is just dropping data points where you know disequilibrium occurred - e.g. wars, etc.

[15]To derive the quantities corresponding to the prices solving the system, it will be necessary to know the "effective" demand and supply functions for all but one market.

[16]Because of "spillover" effects from the markets that are in disequilibrium, the functions involved may not be identical to ex ante functions.

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