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A MODEL OF STOCHASTIC PROCESS SWITCHING

by

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Often, a policy authority such as a central bank operates by establishing a policy rule to set the variables under its control. Such a rule is allowed to operate freely as long as certain endogenous variables of interest to the authority remain within particular bounds; however, when those endogenous variables cross their bounds, the authority switches to a new policy rule which it had prepared to meet this contingency. Since variables such as prices are determined partly by agents' beliefs about future events, agents' behavior injects the probabilities that policy switches will occur at particular future times into current price determination.

In this paper we explore in a formal model the determination of a current exchange rate when future policy regime switches are possible. In order to do this we develop a new aspect of an otherwise standard exchange-rate model; this key component is the probability density function (p.d.f) for the first passage through a barrier of the endogenous variable (the exchange rate) which interests the policy authority.[1/] Since analytical solutions for first passage p.d.f.'s are available for only a limited number of stochastic processes, we are restricted to these processes in formulating our example. However, within this class of processess, our results are generally applicable to many different kinds of macroeconomic problems.

We present our ideas in the context of a model of exchange rate determination. Our choice of a specific example is intended to add concreteness to the analysis but should not be interpreted as setting limits on the applicability of the analysis. Indeed, the structure of the problem at hand virtually duplicates the structures which would be appropriate for studying problems such as a monetary authority's possible return to an interest rate rule, the possible introduction of wage and price controls, possible tax reform, the future fixing of gold price or virtually any other uncertain future policy switch. In addition our set-up should be useful for studying the effects of currently non-binding regulations. For example, an economy with a Regulation Q ceiling on some interest rates may currently be below that ceiling. However, if relevant interest rates were to reach the ceiling then some of the stochastic processes in the economy would change as a result of the now binding regulation.

In the specific example we study, when agents know that at a future time the exchange rate will be fixed at a known level, then in a rational expectations world the solution for the current exchange rate assumes a form reflecting such knowledge. If the level is known while the timing is uncertain, then, though the solution technique is analogous, the acutal solution for the current exchange rate will be a more complex form. As an example of the latter case, our results are particularly applicable to studying the movements of the French and British exchange rates in the early 1920's.

In section I we set up the exchange-rate model when future fixing is possible. In section II we present the major steps necessary to produce an exchange-rate solution.

I). Determining the Current Exchange Rate When Future Fixing is Likely

That the exchange rate between two currencies is allowed to float freely means that governments do not intervene currently in exchange markets to set the rate. However, it is possible that under some future contigencies a government may intervene and establish a fixed rate system; this possibility will partly determine the current floating rate through its effect on expectations.

The specific example that we have in mind is that of Britain in the 1920's. The British decision to return to the gold standard at the pre-war parity of $4.86/£ was announced in the Budget Speech of April 28, 1925. and effective in the exchange market the next day (Moggeridge 1969, p. 9). However, as early as 1918 the Treasury and Ministry of Reconstruction appointed a Committee on Currency under Lord Cunliffe, which reported in 1919 "in our opinion it is imperative that after the war the conditions necessary to the maintenance of an effective gold standard should be restored without delay" (Moggeridge 1969, p. 12). Since the dollar was fixed to gold at that time, the British government was indicating that in the future it would fix the dollar-pound exchange rate at its pre-World War I level; the timing depended on achieving purchasing power parity at the pre-war exchange rate. Adopting such a policy affects the current exchange rate. Here we present a model in which this result is explicit.

In order to highlight the novel aspects of our study we adopt the simplest exchange-rate model popular in the current literature. This is the monetary model of Bilson (1978), Frenkel (1978) and Mussa (1978). The model consists of semi-log linear money demand functions for the countries studied, assumptions of purchasing power parity and uncovered interest parity, and an assumption that semi-elasticities of money demand with respect to interest rates are identical across countries.

The model is described by the following equations

$$m(t) - p(t) = \alpha_0 + \alpha_1 y(t) - \alpha_2 i(t) + v(t); \quad \alpha_1, \alpha_2 > 0 \tag{1}$$

$$\overset{*}{m}(t) - \overset{*}{p}(t) = \overset{*}{\alpha}_0 + \overset{*}{\alpha}_1 \overset{*}{y}(t) - \overset{*}{\alpha}_2 \overset{*}{i}(t) + \overset{*}{v}(t); \quad \overset{*}{\alpha}_1, \overset{*}{\alpha}_2 > 0 \tag{2}$$

$$p(t) = \overset{*}{p}(t) + x(t) \tag{3}$$

$$i(t) = \overset{*}{i}(t) + E(\dot{x}(t) \mid I(t)). \tag{4}$$

Lower case letters generally denote logarithims; an asterisk (*) over a variable denotes "foreign" (U.K.) and

$m(t)$ = money supply

$p(t)$ = price level

$y(t)$ = output

$i(t)$ = interest rate (level)

$v(t)$ = stochastic disturbance

$x(t)$ = exchange rate.

$E(\dot{x}(t) \mid I(t))$ = expected rate of change of $x(t)$ conditional on $I(t)$

$I(t)$ = time t information set containing the structure of the model and all variables dated t or earlier

The left-hand side of (1) is the real domestic money supply which must equal real money demand, the right-hand side of (1). The money demand function is the basic behavioral building block of the model and its parameters. $(\alpha_0, \alpha_1, \alpha_2)$ are assumed to be structural. For simplicity we impose $\alpha_2 = \overset{*}{\alpha}_2$.[2/] Equation (3) is the assumption of purchasing power parity, which is an arbitrage condition in a one-good world. Equation (4) is the condition of uncovered interest parity, which, with risk neutrality, follows from an assumption that domestic and foreign earning assets are perfect substitutes.[3/] We assume that $m(t)$, $\overset{*}{m}(t)$, $y(t)$, $\overset{*}{y}(t)$, $v(t)$ and $\overset{*}{v}(t)$ are exogenous to $x(t)$.

Combine (1) - (4) to obtain

$$m(t) - \overset{*}{m}(t) - x(t) = \alpha_0 - \overset{*}{\alpha}_0 + \alpha_1 y(t) - \overset{*}{\alpha}_1 \overset{*}{y}(t) - \alpha_2 E(\dot{x}(t)|I(t)) + v(t) - \overset{*}{v}(t). \tag{5}$$

We define $K(t) \equiv \overset{*}{\alpha}_0 - \alpha_0 + \overset{*}{\alpha}_1 \overset{*}{y}(t) - \alpha_1 y(t) - \overset{*}{m}(t) + m(t) + \overset{*}{v}(t) - v(t)$. Hence (5) may be written as

$$x(t) = K(t) + \alpha_2 E(\dot{x}(t)|I(t)). \tag{6}$$

Equation (6) is the standard sort of equation that monetary models have produced and is a structural semi-reduced form consistent with a wide variety of models. To address the problem of the future fixing of an exchange rate we must specify both the stochastic nature of the exogenous forcing function $K(t)$ and the nature of the policy rule whereby the monetary authority decides the time for fixing the exchange rate. With rational expectations, the decision to fix the exchange rate implies a decision to change the stochastic nature of $K(t)$. This follows from equation (6); when $x(t)$ is fixed, with rational expectations, $E(\dot{x}(t)|I(t))$ must be zero, hence $K(t)$ must be fixed. For the purposes of this example we will assume that, as long as the monetary authority does not actively fix the exchange rate, $K(t)$ is a random walk with drift, i.e. $K(t)$ can be written as

$$K(t) = K(0) + \eta\, t + e(t) \tag{7}$$

where η is the drift rate and $e(s)$ is a Wiener process, i.e. $e(s) \sim N(0, \sigma^2 s)$. While many alternative specifications for $K(t)$ are possible, we select a process reflecting a U.K. government goal to return to a pre-war parity, fixed exchange rate. Control over the process governing $K(t)$ can be exercised by control of $M^*(t)$ so that $K(t)$ will drift toward the desired fixed exchange rate.

In order to specify a policy rule for when the exchange rate will be fixed, we suppose that the monetary authorities in the foreign country will fix the exchange rate when purchasing power parity holds at some particular $\bar{x}$, i.e. when $\bar{x} = p(t) - \overset{*}{p}(t)$. Since by assumption the domestic price level minus the foreign price level is too low for this to obtain currently, we expect $p(t) - \overset{*}{p}(t) = x(t)$ to make a first passage through $\bar{x}$ from below at the time of the exchange rate's fixing.[4/] At any time t, the moment T in the future at which this first passage occurs is random with a p.d.f. $f(T-t|\bar{x}, K(t))$, which is conditional on $\bar{x}$ and $K(t)$.

Taking expectations of both sides of (6) conditional on the information set $I(t)$ available to agents at time t, we find

$$E(x(t)|I(t)) = E(K(t)|I(t)) + \alpha_2 E(\dot{x}(t)|I(t)) \tag{8}$$

This is a differential equation in the expected exchange rate conditional on $I(t)$; rearranging, we have

$$E(\dot{x}(t)|I(t)) = -\frac{1}{\alpha_2} E(K(t)|I(t)) + \frac{1}{\alpha_2} E(x(t)|I(t)) \tag{9}$$

Given a terminal condition we can solve (9) for the expected (and therefore actual) exchange rate at time t.

Suppose first that purchasing power parity at the exchange rate $\bar{x}$ occurs at time T; then the exchange rate is fixed at $\bar{x}$ for $\tau > T$ and $x(T) = \bar{x}$. Since x(T) is fixed at T, its expected rate of change conditional on fixing at T is zero at T and hence, from (6), $\bar{x} = K(T)$. That $x(\tau)$ makes a first passage through $\bar{x}$ at T is equivalent to $K(\tau)$ making a first passage through $\bar{x}$ at T.[5/]

Conditional on first passage at T, the current exchange rate (and its current expectations) can be determined as

$$E(x(t)|I(t), T) = \bar{x} \exp\{\frac{t - T}{\alpha_2}\} + \frac{1}{\alpha_2} \exp\{\frac{t}{\alpha_2}\} \int_t^T E(K(\tau)|I(t), T) \exp\{-\frac{\tau}{\alpha_2}\}d\tau \qquad (10)$$

where $E(K(\tau)|I(t), T)$ indicates the expected path of $K(\tau)$, $t \leq \tau \leq T$, given I(t) and $K(T) = \bar{x}$ for the first time. The unconditional exchange rate is then the integral of (11) weighted by the first passage p.d.f.

$$x(t) = \int_t^\infty E(x(t)|I(t), T) f(T - t|\bar{x}, K(t))dT \qquad (11)$$

Equation (11) is of the form of a typical solution to a rational expectations model. The problem which remains is to express the right hand side of (11) in terms of a finite number of in principle observable variables. In linear rational expectations models this final step is often accomplished by conjecturing that the solution is a linear function of the state variables and then requiring the unknown coefficients in the conjectured solution to obey the model at hand. This is the method of undetermined coefficients recently popularized by Lucas (1972). Our problem, however, is substantially more difficult because the as yet unknown non-linear form of the solution must be constructed from first principles. The solution we seek will be a non-structural relation whose parameters will depend on current and expected future government behavior.

II) The Forms of $f(T - t|\bar{x}, K(t))$ and $E(K(\tau)|I(t), T)$

To obtain the reduced form exchange-rate equation we proceed in two steps, first finding the density function $f(T-t|\bar{x}, K(t))$ and second finding $E(K(\tau)|I(t), T)$.[6/] Analytical expressions for these two magnitudes may then be substituted into (10) and (11), yielding the reduced form we seek.

The solution for the first passage p.d.f. of a Wiener process with drift is available in standard texts (see Karlin and Taylor, p. 363). The p.d.f. over the first passage of $K(\tau)$ through $\bar{x}$, given $K(t) < \bar{x}$ is

$$f(T - t|\bar{x}, K(t)) = \frac{\bar{x} - K(t)}{\sigma\sqrt{2\pi}(T - t)^{3/2}} \exp\{-\frac{1}{2}\frac{[\bar{x} - K(t) - \eta(T - t)]^2}{\sigma^2(T - t)}\} \qquad (12)$$

The derivation of $E(K(\tau)|I(t), T)$ is an exercise in stochastic processes. We first write out the explicit formula for this expectation and explain its components. Then we explain the determination of $E(K(\tau)|I(t), T)$.

Recall that $E(K(\tau)|I(t), T, t \leq \tau \leq T)$ is the expectation of $K(\tau)$ given $K(t)$ and given that at time T, $K(T) = \bar{x}$ for the first time. Let $T_1 \equiv T - t$, $\tau_1 \equiv \tau - t$, $Z \equiv \bar{x} - K(t)$ and $\overset{*}{Z} \equiv \frac{Z}{\sigma}\sqrt{\frac{1 - \tau_1/T_1}{\tau_1}}$. Then the explicit formula for the conditional expectation is.

$$E(K(\tau)|I(t), T) = \bar{x} - C_2/C_1 \qquad (13)$$

where

$$C_2 \equiv [1 - \frac{\tau_1}{T_1}]\{[(1 - \frac{\tau_1}{T_1})Z^2 + \sigma^2\tau_1]\Phi(\overset{*}{Z}) + \sigma Z\tau_1^{\frac{1}{2}}(1 - \frac{\tau_1}{T_1})^{\frac{1}{2}}\phi(-\overset{*}{Z})$$

$$- \exp\{(1 - \frac{\tau_1}{T_1})\frac{2\eta Z}{\sigma^2}\}[(1 - \frac{\tau_1}{T_1})Z^2 + \sigma^2\tau_1]\Phi(-\overset{*}{Z}) - \sigma Z\tau_1^{\frac{1}{2}}(1 - \frac{\tau_1}{T_1})^{\frac{1}{2}}\phi(-\overset{*}{Z})]\} \tag{14}$$

and

$$C_1 \equiv [\sigma\tau_1^{\frac{1}{2}}(1 - \frac{\tau_1}{T_1})^{\frac{1}{2}}\phi(\overset{*}{Z}) + Z(1 - \frac{\tau_1}{T_1})\Phi(\overset{*}{Z})]$$

$$- \exp\{(1 - \frac{\tau_1}{T_1})\frac{2\eta Z}{\sigma^2}\}\ [\sigma\tau_1^{\frac{1}{2}}(1 - \frac{\tau_1}{T_1})^{\frac{1}{2}}\phi(\overset{*}{Z}) - Z(1 - \frac{\tau_1}{T_1})\Phi(-\overset{*}{Z})]\ . \tag{15}$$

In these formulas, $\phi(x) \equiv \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}$ and $\Phi(x) \equiv \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\}dy$.

To derive formulas (13)-(15), we must find the conditional density of $K(\tau)$, given T, the time of first passage through $\bar{x}$, where $T > \tau > t$. Call this density function $h(K(\tau)|T)$. Then we need only multiply by $K(\tau)$ and integrate to determine the first moment. We can find this density function by first determining the joint density over $(K(\tau), T)$. For simplicity, let us assume that we are looking forward from time $t = 0$ and that $K(0) = 0$. (These assumptions are relaxed in our reported results; see Appendix section 4.)

The joint density function equals the conditional density function over $K(\tau)$ multiplied by the marginal density function over T, $f(T)$, i.e.

$$g(K(\tau), T) = h(K(\tau)|T)f(T). \tag{16}$$

The joint density also equals the conditional density over T, given $K(\tau)$, which we denote by $F(T|K(\tau))$, multiplied by the marginal density over $K(\tau)$, $H(K(\tau))$, i.e.

$$g(K(\tau), T) = F(T|K(\tau))H(K(\tau)) \tag{17}$$

$H(K(\tau))$ does not depend on the time $T > \tau$ at which first passage occurs, but it is conditional on first passage not having occurred prior to τ.

Since they are relatively easy to derive, we will use the density functions in (17) to construct the joint density $g(K(\tau), T)$; then we can determine the conditional density $h(K(\tau)|T)$ from (16). First we develop $H(K(\tau))$.

Since $K(\tau)$ is a Wiener process with drift and with a starting value $K(0) = 0$, $K(\tau)$ has an unconditional p.d.f.

$$G(K(\tau)) = \frac{1}{\sqrt{2\pi\tau}\sigma} \exp\{-\tfrac{1}{2}\frac{(K(\tau) - \eta\tau)^2}{\sigma^2 t}\} = \frac{1}{\sigma\sqrt{\tau}}\phi(\frac{K(\tau) - \eta\tau}{\sigma\sqrt{\tau}}) \tag{18}$$

where $\phi(x) \equiv \frac{1}{\sqrt{2\pi}}\exp\{-\tfrac{1}{2}x^2\}$(See Karlin and Taylor, p. 356). $H(K(\tau))$ is the density of $K(\tau)$ <u>conditional</u> on $K(\tau)$'s having remained below $\bar{x}$ prior to τ. Thus, from the probability weight given to $K(\tau)$ by $G(K(\tau))$, we must subtract the weight associated with all paths (like 1 in Figure 1) which pass through $K(\tau)$ at τ but which also have passed through $\bar{x}$ prior to τ.

There are an infinity of paths which, like path 1, pass through $\bar{x}$ for the first time at t_1 and through $K(\tau)$ at time τ. Given that a path starts at $\bar{x}$ at t_1, the unconditional probability weight associated with its passing through $K(\tau)$ at time τ is obtained from (18) as:

$$G(K(\tau)|K(t_1) = \bar{x})) = \frac{1}{\sigma\sqrt{\tau - t_1}}\phi(\frac{K(\tau) - \bar{x} - \eta(\tau - t_1)}{\sigma\sqrt{\tau - t_1}}) \tag{19}$$

The probability weight associated with a path's passing through $\bar{x}$ for the first time at t_1 is, (from equation (12),

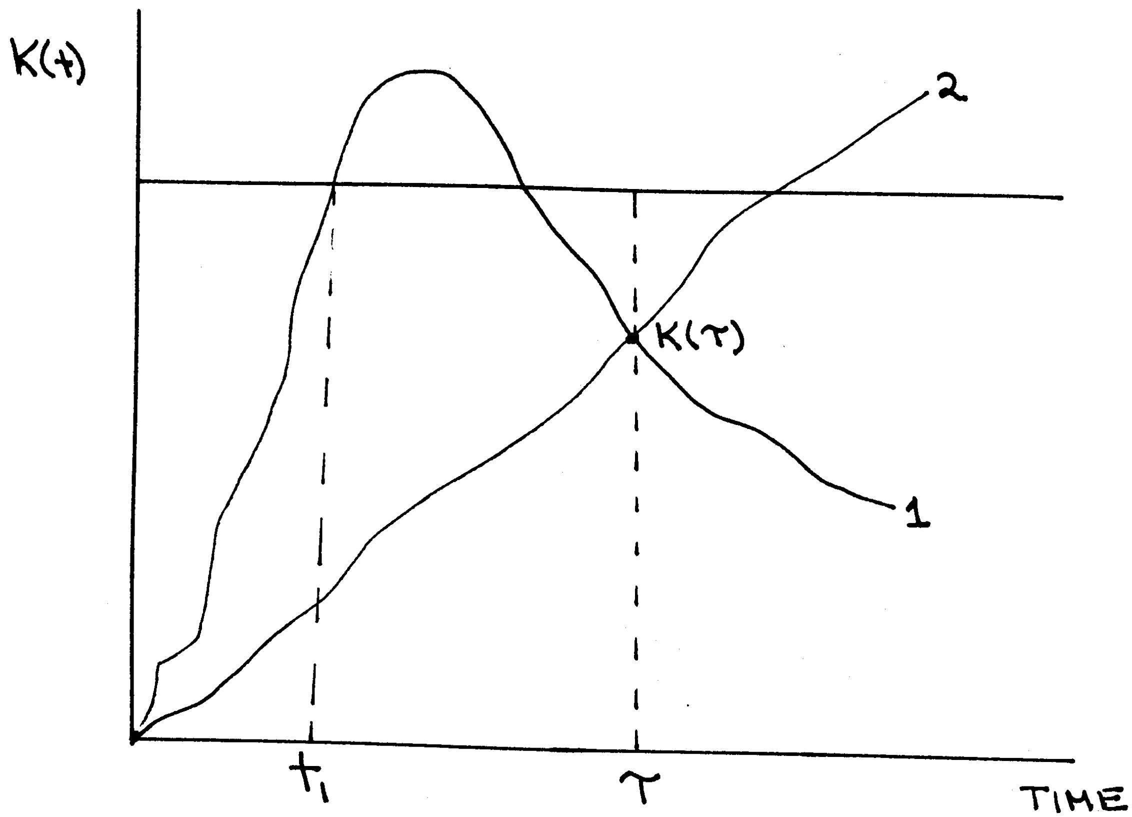

FIGURE 1

$$f(t_1|K(0) = 0) = \frac{\bar{x}}{\sigma\sqrt{2\pi}\, t_1^{3/2}} \exp\{-\frac{(\bar{x} - \eta t_1)^2}{2\sigma^2 t_1}\} = \frac{\bar{x}}{\sigma t_1^{3/2}} \phi(\frac{\bar{x} - \eta t_1}{\sigma\sqrt{t_1}}) \quad (20)$$

Then the probability weight associated with the set of all paths which both pass through $\bar{x}$ for the first time at t_1 and pass through $K(\tau)$ at τ is

$$G(K(\tau)|K(t_1) = \bar{x}) f(t_1|K(0) = 0) =$$

$$\frac{\bar{x}}{\sigma t_1^{3/2}} \phi(\frac{\bar{x} - \eta t_1}{\sigma\sqrt{t_1}}) \frac{1}{\sigma\sqrt{\tau - t_1}} \phi(\frac{K(\tau) - \bar{x} - \eta(\tau - t_1)}{\sigma\sqrt{\tau - t_1}}) \quad (21)$$

Therefore, the probability weight associated with <u>all</u> paths which both pass through $\bar{x}$ at some time prior to τ and equal $K(\tau)$ at τ is the integral over t_1 of (21):

$$\int_0^\tau \frac{\bar{x}}{\sigma t_1^{3/2}} \phi(\frac{\bar{x} - \eta t_1}{\sigma\sqrt{t_1}}) \frac{1}{\sigma\sqrt{\tau - t_1}} \phi(\frac{K(\tau) - \bar{x} - \eta(\tau - t_1)}{\sigma\sqrt{\tau - t_1}}) dt_1 \quad (22)$$

To determine $H(K(\tau))$ up to a normalizing constant we need only subtract (22) from (18)

$$H(K(\tau)) = \frac{1}{\sigma\sqrt{\tau}} \phi(\frac{K(\tau) - \eta\tau}{\sigma\sqrt{\tau}}) -$$

$$\int_0^\tau \frac{\bar{x}}{\sigma t_1^{3/2}} \phi(\frac{\bar{x} - \eta t_1}{\sigma\sqrt{t_1}}) \frac{1}{\sigma\sqrt{\tau - t_1}} \phi(\frac{K(\tau) - \bar{x} - \eta(\tau - t_1)}{\sigma\sqrt{\tau - t_1}}) dt_1 \quad (23)$$

To determine the joint p.d.f. $g(K(\tau), T)$ we must multiply $H(K(\tau))$ by the conditional p.d.f. $F(T|K(\tau))$. But again from equation (12),

$$F(T|K(\tau)) = \frac{\bar{x} - K(\tau)}{\sigma\sqrt{2\pi}(T - \tau)^{3/2}} \exp\{-\frac{(\bar{x} - K(\tau) - \eta(T - \tau))^2}{2\sigma(T - \tau)}\}$$

$$= \frac{\bar{x} - K(\tau)}{\sigma(T - \tau)^{3/2}} \phi(\frac{\bar{x} - K(\tau) - \eta(T - \tau)}{\sigma\sqrt{T - \tau}}). \quad (24)$$

Finally,

$$g(K(\tau), T) = CH(K(\tau))F(T|K(\tau)) \quad (25)$$

where C is a normalizing constant. $h(K(\tau), T)$ is simply (25) divided by $f(T)$ and evaluated at a particular value of T.

To derive C_1 and C_2, we performed a change of variable in (25) to produce a p.d.f. over $u(\tau) \equiv \bar{x} - K(\tau)$. In the formula (13), C_1 is simply the inverse of the normalizing constant for this p.d.f. while C_2 is the unnormalized first moment of this p.d.f. Hence, $C_2/C_1 = E(u(\tau)|u(t), T)$ so that $E(K(\tau)|K(t), T) = \bar{x} - C_2/C_1$. Deriving the actual formulas (14) - (15) requires the cranking out of some horrendous integrals, which we relegate to the appendix.

III) Application

We have derived analytical expressions for $f(T - t|\bar{x}, K(t))$ and $E(K(\tau)|K(t), T)$. The next step is to substitute these results into (10) and (11) and continue the integration. However, the remaining double integral has proven intractable to us, so we simply report our solution for $x(t)$ as

$$x(t) = \int_t^\infty [\bar{x} \exp\{\frac{t - T}{\alpha_2}\} + \frac{1}{\alpha_2} \exp\{\frac{t}{\alpha_2}\} \int_t^T (\bar{x} - C_2(\tau)/C_1(\tau) \exp\{-\frac{\tau}{\alpha_2}\}d\tau]$$

$$\cdot f(T - t|\bar{x}, K(t))dT \quad (26)$$

The nonlinear exchange-rate equation resulting from the above can in principle be estimated using a combination of nonlinear techniques and numerical integration sub-routines.

The unfortunate feature of our result is that it implies that it is not appropriate to estimate on exchange-rate equation by typical linear methods during a period when agents are anticipating stochastic process switching. For example, Frenkel and Clements (1980) estimate a US/UK exchange-rate equation over the period February 1921 to May 1925, which encompasse a large part of the period when agents may have been anticipating process switching. To allow for the endogeneity of interest rate differentials Frenkel and Clements used a linear two stage least squares procedure. According to our results the first stage of their procedures should have been specified in accord with our non-linear exchange rate equation.

It seems to us that the problem encountered in Frenkel and Clements may be quite widespread. Indeed, whenever policy makers deliberate, they inject into agents' forecasting problems an element of stochastic process switching. However, it is atypical of such deliberations that they result in a stochastic process switching problem as clearly defined as the British return to pre-war parity.

Footnotes

1/ Models of pricing for some types of options make use of first passage probability density functions. For example, Ingersoll (1977) uses first passage profit's in studying the prices of convertible securities.

2/ By assuming $\alpha_2 = \overset{*}{\alpha}_2$ we are able to determine x(t) without modeling the goods market. Alternatively we could allow $\alpha_2 \gtrless \overset{*}{\alpha}_2$, impose world goods market equilibrium, and produce an exchange-rate solution slightly different from that reported below.

3/ Some empirical support for the assumption of open interest parity can be found in Hansen and Hodrick (1980).

4/ In our example we are treating the U.S. as the home country and the U.K. as the foreign country so $\bar{x} = \ell n(\$4.86/£)$.

5/ The nature of the exchange-rate fixing policy precludes the existence of a multiple solution type bubble which would cause x(t) to rise through $\bar{x}$. However, it does not preclude the existence of negative bubbles which would prevent x(t) from passing through $\bar{x}$ from below even though K(t) passes through $\bar{x}$. Therefore, for the stated equivalence to hold we must explicitly rule out the existence of multiple solutions to (9) of the speculative bubble variety. Hence, the solution to (9) depends only on market fundamentals, as the formal expressions (10) and (11) for a solution explicitly indicate.

6/ We are extremely grateful to J.H. Kemperman for showing us how to derive the conditional expectation of $K(\tau)$.

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Appendix

Derivation of C_1 and C_2

1) Solution to Integral in Text Equation (23)

Notice that the integral part of the right-hand side of equation (23) [text] is a convolution. It is

$$\int_0^{\tau} w(t_1)Z(\tau - t_1)dt_1 \tag{A1}$$

where

$$w(t_1) \equiv \frac{\bar{x}}{\sigma t_1^{3/2}} \phi\left(\frac{\bar{x} - \eta t_1}{\sigma t_1^{1/2}}\right) \tag{A2}$$

and

$$Z(\tau - t_1) \equiv \frac{(\tau - t_1)^{-1/2}}{\sigma} \phi\left(\frac{K(\tau) - \bar{x} - \eta(\tau - t_1)}{\sigma(\tau - t_1)^{1/2}}\right) \tag{A3}$$

It is a property of the Laplace transform, L[ ], that $L[w(\tau)] \cdot L[Z(\tau)] = L[\int_0^{\tau} w(t_1)Z(\tau - t_1)dt_1]$ (see Simmons, pp. 407-408). For our problem

$$w(\tau) = \frac{\bar{x}}{\sigma \tau^{3/2}} \phi\left(\frac{\bar{x} - \eta\tau}{\sigma\tau^{1/2}}\right) \tag{A4}$$

and

$$Z(\tau) = \frac{\tau^{-1/2}}{\sigma} \phi\left(\frac{K(\tau) - \bar{x} - \eta\tau}{\sigma\tau^{1/2}}\right) \tag{A5}$$

This property is useful to us because the problem of integrating (A1) may be stated equivalently as finding $L^{-1}[L[w(\tau)]L[Z(\tau)]]$, where $L^{-1}[\quad]$

is the inverse Laplace transform. We will develop our analytic expression for (A1) using Laplace transforms.

Our first step is to note

$$L[w(\tau)] = \exp\{\frac{\bar{x}}{\sigma^2}(\eta - [\eta^2 + 2\sigma^2 p]^{1/2})\}, \tag{A6}$$

where p is the parameter of the Laplace transform (see Karlin and Taylor, p. 362). Further, since

$$L[\tau w(\tau)] = -\partial L[w(\tau)]/\partial p \tag{A7}$$

(see Simmons, p. 402) we have

$$L[\tau w(\tau)] = \bar{x}[\eta^2 + 2\sigma^2 p]^{-1/2}\exp\{\frac{\bar{x}}{\sigma^2}(\eta - [\eta^2 + 2\sigma^2 p]^{1/2})\}. \tag{A8}$$

(A8) will prove useful in finding $L[Z(\tau)]$, to which we now turn.

Recall that $\phi(x) = \phi(-x)$ so

$$Z(\tau) = \frac{\tau^{-1/2}}{\sigma}\,\phi(\frac{\bar{x} - K(\tau) + \eta\tau}{\sigma\tau^{1/2}}) , \tag{A9}$$

which is the form of $Z(\tau)$ we will work with subsequently. Comparing (A9) with (A4) we note that $Z(\tau)$ is $w(\tau)\tau/\bar{x}$ with the constant in the numerator of $\phi(\ )$ in $Z(\tau)$ being $\bar{x} - K(\tau)$ instead of $\bar{x}$ in the corresponding term in $w(\tau)$ and $-\eta$ in $w(\tau)$ being η in $Z(\tau)$. It follows that we obtain the Laplace transform of $Z(\tau)$ by dividing (A8) by $\bar{x}$ and then replacing $\bar{x}$ in that result with $\bar{x} - K(\tau)$ and replacing $-\eta$ with η (ie, change the sign of η). We obtain

$$L[Z(\tau)] = [\eta^2 + 2\sigma^2 p]^{-1/2}\exp\{\frac{\bar{x} - K(\tau)}{\sigma^2}(-\eta - [\eta^2 + 2\sigma^2 p]^{1/2})\} \tag{A10}$$

From (A6) and (A10) obtain

$$L[w(\tau)]L[Z(\tau)] = [\eta^2 + 2\sigma^2 p]^{-1/2} \exp\{\frac{-2\bar{x} + K(\tau)}{\sigma^2}[\eta^2 + 2\sigma^2 p]^{1/2} + \frac{\eta K(\tau)}{\sigma^2}\} \qquad \text{(A11)}$$

(A11) is the Laplace transform of the integral we seek so we are now looking for the inverse Laplace transform of (A11).

Notice that if in (A10) we replace $K(\tau)$ with $K(\tau) - \bar{x}$ then we will produce the expression on the right hand side of (A11) multiplied by the factor $\exp\{-2\eta\bar{x}/\sigma^2\}$. Thus, we create the function

$$q(\tau) \equiv \exp\{2\eta\bar{x}/\sigma^2\} \frac{\tau^{-1/2}}{\sigma^2} \phi(\frac{K(\tau) - 2\bar{x} - \eta\tau}{\sigma\tau^{1/2}}) \qquad \text{(A12)}$$

and by construction we know $L[q(\tau)] = L[w(\tau)]L[Z(\tau)]$. Hence

$$q(\tau) = L^{-1}[L[w(\tau)]L[Z(\tau)]] = \int_0^\tau w(t_1)Z(\tau - t_1)dt_1,$$

so $q(\tau)$ is the analytic integral we have sought.

We were attempting to solve the integral in text equation (23) so that we could obtain an analytic expression for $g(K(\tau), T)$. To obtain this expression we now use (A12) in text equation (23) and we use equations (23) and (24) in (25) yielding

$$g(K(\tau), T) = C \frac{\tau^{-1/2}}{\sigma}[\phi(\frac{K(\tau) - \eta\tau}{\sigma\tau^{1/2}}) - \exp\{\frac{2\eta\bar{x}}{\sigma^2}\}\phi(\frac{K(\tau) - 2\bar{x} - \eta\tau}{\sigma\tau^{1/2}})]$$

$$\cdot[\frac{(\bar{x} - K(\tau))}{\sigma(T - \tau)^{3/2}}]\phi(\frac{\bar{x} - K(\tau) - \eta(T - \tau)}{\sigma(T - \tau)^{1/2}})]. \qquad \text{(A13)}$$

C is a normalizing constant. Given T, the first time K( ) passes through

$\overline{x}$, (A13) is also the form of the conditional p.d.f. $h(K(\tau)|T)$, except that the normalizing constant for $h(K(\tau)|T)$ will be different because of the need to divide (A13) through by the value of the marginal p.d.f. over T.

2. Deriving the normalizing constant for $h(K(\tau)|T)$

Defining $u \equiv \overline{x} - K(\tau)$, substituting u into (A13) and dropping all the constant coefficients, the function (A13) can be written as

$$u[\phi(\frac{\overline{x} - u - \eta\tau}{\sigma\sqrt{\tau}}) - \exp\{\frac{2\eta\overline{x}}{\sigma^2}\}\phi(\frac{-u - \overline{x} - \eta\tau}{\sigma\sqrt{\tau}})]\phi(\frac{u - \eta(T - \tau)}{\sigma\sqrt{T - \tau}}) \equiv u\Omega(u). \tag{A14}$$

Except for a normalizing constant, (7) is a p.d.f. over u. We are interested in the moments

$$C_n = \int_0^\infty u^n\, \Omega(u)du. \tag{A15}$$

Here $1/C_1$ is the normalizing constant and C_2/C_1 is the conditional mean of u. The conditional mean of $K(\tau) = \overline{x} - C_2/C_1$.

By elementary algebra,

$$\phi(\frac{u - a}{\sigma\tau^{1/2}})\phi(\frac{u - b}{\sigma(\tau - T)^{1/2}}) = (2\pi)^{-1}\exp[\frac{-\{u - (1 - \tau/T)a - (\tau/T)b\}^2}{2\sigma^2\tau(1 - \tau/T)} - \frac{a^2 + b^2}{2\sigma^2 T}] \tag{A16}$$

Using this result with $a = \pm\overline{x} - \eta\tau$ and $b = \eta(T - \tau)$ and remembering that $\phi(r) = \phi(-r)$, we find that C_n is proportional by a factor independent of n to

$$C'_n = \int_0^\infty u^n[\phi(\frac{u - (1 - \tau/T)\overline{x}}{\sigma[\tau(1 - \tau/T)]^{1/2}}) - \exp\{\frac{(1 - \tau/T)2\eta\overline{x}}{\sigma^2}\}\phi(\frac{u + (1 - \tau/T)\overline{x}}{\sigma[\tau(1 - \tau/T)]^{1/2}})]du \tag{A17}$$

To find the normalizing constant we set n = 1 in (A17) and we perform a change of variables using the following definitions

$$\varepsilon_1 \equiv \frac{u - (1 - \tau/T)\bar{x}}{\sigma[\tau(1 - \tau/T)]^{1/2}} \tag{A18}$$

$$\varepsilon_2 \equiv \frac{u + (1 - \tau/T)\bar{x}}{\sigma[\tau(1 - \tau/T)]^{1/2}} \quad . \tag{A19}$$

We have

$$C_1' = \int_{-x^*}^{\infty} [(1 - \tau/T)\bar{x} + \varepsilon_1\sigma\, [\tau(1 - \tau/T)]^{1/2}]\sigma[\tau(1 - \tau/T)]^{1/2}\phi(\varepsilon_1)d\varepsilon_1$$

$$- \exp\{\frac{(1 - \tau/T)2\eta\bar{x}}{\sigma^2}\int_{x^*}^{\infty}[-(1 - \tau/T)\bar{x} + \varepsilon_2\sigma[\tau(1 - \tau/T)]^{1/2}]\sigma[\tau(1 - \tau/T)]^{1/2}\cdot\phi(\varepsilon_2)d\varepsilon_2 \tag{A20}$$

where

$$x^* \equiv \frac{\bar{x}(1 - \tau/T)^{1/2}}{\sigma\tau^{1/2}}$$

Recall that $\phi(w) = (2\pi)^{-1/2}\exp\{-1/2w^2\}$ and define $\Phi(w) \equiv \int_{-\infty}^{w}\phi(u)du$ so $\Phi(w) + \Phi(-w) = 1$. (A20) reduces to

$$C_1' = \sigma\tau^{1/2}(1 - \tau/T)^{3/2}\Phi(x^*) + (2\pi)^{-1/2}\sigma^2\tau(1 - \tau/T)\exp\{-1/2x^{*2}\}$$

$$- \exp\{\frac{(1 - \tau/T)2\eta\bar{x}}{\sigma^2}\}[-\sigma\bar{x}\tau^{1/2}(1 - \tau/T)^{3/2}\Phi(-x^*) + (2\pi)^{-1/2}\sigma^2$$

$$\cdot\tau(1 - \tau/T)\exp\{-1/2x^{*2}\}] \tag{A21}$$

3. Deriving the C_2' Integral

Now consider

$$C_2' = \int_{-x^*}^{\infty} [(1 - \tau/T)\bar{x} + \varepsilon_1\sigma[\tau(1 - \tau/T)]^{1/2}]^2\sigma[\tau(1 - \tau/T)]^{1/2}\phi(\varepsilon_1)d\varepsilon_1$$

$$- \exp\{\frac{(1 - \tau/T)2\eta\bar{x}}{\sigma^2}\} \int_{x^*}^{\infty} [-(1 - \tau/T)\bar{x} + \varepsilon_2\sigma[\tau(1 - \tau/T)]^{1/2}]^2\sigma[\tau(1 - \tau/T)]^{1/2}$$

$$\phi(\varepsilon_2)d\varepsilon_2 \qquad \text{(A22)}$$

which results from (A17) with n = 2 and ε_1, ε_2 and x^* defined as before.

The right-hand side of (A22) is now broken into two integrals and we will evaluate these in turn.

3a. First Integral

The first integral in (A22) is

$$\sigma[\tau(1 - \tau/T)]^{1/2} \int_{-x^*}^{\infty} [(1 - \tau/T)^2\bar{x}^2 + 2\sigma\bar{x}\tau^{1/2}(1 - \tau/T)^{3/2}\varepsilon_1$$

$$+ \sigma^2\tau(1 - \tau/T)\varepsilon_1^2](2\pi)^{-1/2}\exp\{-1/2\varepsilon_1^2\}d\varepsilon_1 \qquad \text{(A23)}$$

where we have substituted $\phi(\varepsilon_1)$'s functional form and expanded the quadratic from the first part of (A22). The term in square brackets under the integral in (A23) is a sum of three elements and we will treat these in turn.

3ai. $$\int_{-x^*}^{\infty} \bar{x}^2(1 - \tau/T)^2(2\pi)^{-1/2}\exp\{-1/2\varepsilon_1^2\}d\varepsilon_1 = (1 - \tau/T)^2\bar{x}^2\Phi(x^*) \qquad \text{(A24)}$$

3aii. $\int_{-x^*}^{\infty} 2\sigma\bar{x}\tau^{1/2}(1 - \tau/T)^{3/2}(2\pi)^{-1/2}\varepsilon_1 \exp\{-1/2\varepsilon_1^2\}d\varepsilon_1 = 2\sigma\bar{x}\tau^{1/2}(1 - \tau/T)^{3/2}$

$$\cdot\phi(-x^*) \tag{A25}$$

3aiii. The third term is

$$\int_{-x^*}^{\infty} \sigma^2\tau(1 - \tau/T)\varepsilon_1^2(2\pi)^{-1/2}\exp\{-1/2\varepsilon^2\}d\varepsilon_1 \tag{A26}$$

and we must integrate this by parts. Set $dF = (2\pi)^{-1/2}\varepsilon_1 \exp\{-1/2\varepsilon_1^2\}d\varepsilon_1$ and set $H = \varepsilon_1$. We know $\int HdF = HF - \int FdH$. Hence $\int HdF = -[(2\pi)^{-1/2}\varepsilon_1 \exp\{-1/2\varepsilon_1^2\}]_{-x^*}^{\infty} - \int_{-x^*}^{\infty} -(2\pi)^{-1/2} \exp\{-1/2\varepsilon_1^2\}d\varepsilon_1$

or

$$\int HdF = -x^*\phi(-x^*) + \Phi(x^*)$$

Since $\int HdF$ is (A25) up to a constant we find that (A25) is

$$\sigma^2\tau(1 - \tau/T)[\Phi(x^*) - x^*\phi(-x^*)] \ . \tag{A27}$$

Summarizing, the first integral on the right hand side of (A22), which is (A23), is

$$\sigma[\tau(1 - \tau/T)]^{1/2}[(1 - \tau/T)^2 \bar{x}^2\Phi(x^*) + 2\sigma\bar{x}\tau^{1/2}(1 - \tau/T)^{3/2}\phi(-x^*)$$

$$+ \sigma^2\tau(1 - \tau/T)[\Phi(x^*) - x^*\phi(x^*)]] \tag{A28}$$

When we substitue $x^* = \frac{\bar{x}(1 - \tau/T)^{1/2}\tau^{-1/2}}{\sigma}$ for the x* coefficient in the last term of (A28) we obtain

$$\sigma\tau^{1/2}(1 - \tau/T)^{3/2}[\{(1 - \tau/T)\bar{x}^2 + \sigma^2\tau\}\Phi(x^*) + \sigma\bar{x}\tau^{1/2}(1 - \tau/T)^{1/2}\phi(-x^*)], \quad (A29)$$

and this is our expression for (A23).

3b. Second Integral

The second integral in (A22) is $-\exp\{(1 - \tau/T)2\eta\bar{x}\sigma^{-2}\}\sigma\tau^{1/2}(1 - \tau/T)^{1/2}$.

$$\int_{x^*}^{\infty} [(1 - \tau/T)^2 \bar{x}^2 - 2\bar{x}(1 - \tau/T)\sigma\tau^{1/2}\varepsilon_2 + \sigma^2\tau(1 - \tau/T)\varepsilon_2^2]\phi(\varepsilon_2)d\varepsilon_2 \quad (A30)$$

Substituting in (A30) the results in (A24), (A25) and (A28) the second integral equals

$$-\exp\{(1 - \tau/T)2\eta\bar{x}\sigma^{-2}\}\sigma\tau^{1/2}(1 - \tau/T)^{1/2}[(1 - \tau/T)^2\bar{x}^2\Phi(-x^*)$$

$$-2\sigma\bar{x}\tau^{1/2}(1 - \tau/T)^{3/2}\phi(-x^*) + \sigma^2\tau(1 - \tau/T)(\Phi(-x^*) + x^*\phi(x^*))] \quad (A31)$$

Terms in (A31) may be rearranged to give

$$-\exp\{(1 - \tau/T)2\eta\bar{x}\sigma^{-2}\}\sigma\tau^{1/2}(1 - \tau/T)^{1/2}[(1 - \tau/T)^2\bar{x}^2\Phi(-x^*)$$

$$-\bar{x}\sigma\tau^{1/2}(1 - \tau/T)^{3/2}\phi(-x^*) + \sigma^2\tau(1 - \tau/T)\Phi(-x^*)] \quad (A32)$$

Combining (A32) and (A29) we find

$$C_2' = \sigma\tau^{1/2}(1 - \tau/T)^{3/2}\{[(1 - \tau/T)\bar{x}^2 + \sigma^2\tau]\Phi(x^*) + \bar{x}\tau^{1/2}(1 - \tau/T)^{1/2}\sigma\phi(-x^*)$$

$$-\exp\{(1 - \tau/T)2\eta\bar{x}\sigma^{-2}\}[(1 - \tau/T)\bar{x}^2\Phi(-x^*) + \sigma^2\tau\Phi(-x^*) - \sigma\bar{x}\tau^{1/2}(1 - \tau/T)^{1/2}\phi(-x^*)]\}$$

4. Alterations needed to Produce the Form reported in the Text

Since we are interested only in the ratio C_2'/C_1' we can remove all coefficients common to the terms in C_1' and C_2'. Since $\sigma\tau^{\frac{1}{2}}(1 - \tau/T)^{\frac{1}{2}}$ is common to both C_1' and C_2' it is not included in the values which we report for C_1 and C_2 in the text.

Notice also that the text uses for notation τ_1, T_1, Z_1 and Z^* in place of τ, T, $\bar{x}$ and x^*, respectively, which we have used in the appendix. Recall that for simplicity we assumed that the time at which this forecast is made is time zero for the derivations in the appendix. The time for which the forecast is made is called τ in the appendix. In the text the time at which the forecast is made is called t; the time for which the forecast is made is called τ. Hence the variable $\tau_1 \equiv \tau - t$, in the text notation, is substituted for τ, in the notation of the appendix. Similarly, $T_1 \equiv T - t$ in the text notation is substituted for T in the notation of the appendix. In the appendix K(0) is set at zero; in the text K(t), the value of K( ) at the time at which the forecast is made, need not be zero. Hence, we subtract K(t) from $\bar{x}$ to derive a barrier equivalent to $\bar{x}$ in the text. Defining $Z \equiv \bar{x} - K(t)$ we substitute Z for $\bar{x}$ in the notation of the appendix. Finally, letting $Z^* \equiv \frac{Z}{\sigma} \frac{1 - \tau_1/T_1}{\tau_1}$, we substitute Z^* for x^*. This produces the formulas for C_2 and C_1 in the text.