International Finance Discussion Papers

Number 289

July 1986

INCOMPLETE INSURANCE, IRREVERSIBLE INVESTMENT, AND THE MICROFOUNDATIONS OF FINANCIAL INTERMEDIATION

by

Robert A. Johnson

ABSTRACT

The financial intermediary is shown to result from a market imperfection related to the costly monitoring of the actions of consumers. In such an environment complete insurance is not obtainable and consumers respond by holding some of their wealth as precautionary balances in order to self-insure. Precautionary balances are those financial vehicles which permit one to invest and then liquidate with the smallest amount of loss because of the "sunk costs" associated with the transaction. An economy of N identical consumers is created and it is shown that a financial intermediary which collects the precautionary balances of the community can then implement risk sharing and liberate social resources for greater investment.

Incomplete Insurance, Irreversible Investment, and the Microfoundations of Financial Intermediation

by

Robert A. Johnson*

I. Introduction

Economists have long recognized that financial intermediation is anomalous in a perfectly competitive economy with complete markets. Recent efforts to provide an explanation for the existence of financial intermediaries place great emphasis on imperfections in the market and proceed to illustrate the role that indirect finance plays in improving the market allocation in the presence of such imperfections.

The literature created as a result of this research strategy can be conceptually divided into two groups corresponding to the roles played by the financial intermediary in improving allocative efficiency.[1] The first group models the financial institution as a broker-dealer which, due to its ability to exploit information or transactions costs, can lower the cost of borrowing and/or increase the return to lending when compared with direct interaction between borrowers and lenders. The second group assumes, often implicitly, that insurance markets are incomplete and exhibit the economies achieved by the financial intermediary due to its ability to consolidate and transform risk. It is this second aspect of financial intermediation, the transformation of risk, that is the subject of this paper.

The model developed in the succeeding sections portrays the financial intermediary as a bank which receives the precautionary balances of a community of consumers (demand deposits) and invests some portion of its holdings in interest-bearing assets which are not redeemable prior to a specified date in the future.

The consumers divide their initial wealth between liquid balances held in a deposit account at the bank and direct investment in the interest-bearing security which is completely irreversible (cannot be redeemed prior to maturity). The consumers derive utility from holding deposits because they provide self insurance against the possibility of a "rainy day" (a low stochastic income realization) in the future.

The consumer's decision to hold precautionary balances is a response to the lack of complete insurance markets.[2] Were competitive complete insurance available, the consumer could write a set of contracts contingent on his income realizations and precautionary balances would be unnecessary. In the sections that follow, I assume that the non-availability of complete insurance derives from the structure of information. I assume that individual stochastic income realizations are private information and that it is prohibitively costly for any potential insurer to ascertain the individual's particular income state.

In an environment characterized by such asymmetries in information, the individual holds some *ex ante* optimal level of precautionary balances and withdraws some of them in response to bad income realizations. Provided that individual risks are not perfectly correlated across consumers, the variability of aggregate bank deposits per consumer is less than individual deposit variability. The bank, by collecting consumers' precautionary balances, in effect implements a risk sharing arrangement. The pooling of consumers' idiosyncratic withdrawal demands permits the bank to economize on social resources devoted to liquidity and therefore to increase the proportion of social wealth which can be devoted to investment for any given rate of return.

In the sections below, I present an analytic representation of the ideas outlined above. Section II constructs a model of the individual consumer who derives utility from consumption in each of two periods and who must divide his initial wealth between precautionary balances and investment in an irreversible asset prior to observing income, which is stochastic. From this, one can derive the consumer's schedule of optimal deposit withdrawals as a function of income and then, given the distribution function governing income uncertainty, the distribution of individual withdrawals.

Section III develops the model of the banking firm. First, the distribution of aggregate deposit withdrawals is derived from the distribution of individual withdrawals for any economy of N consumers with identically and independently distributed income. Then the bank, with the knowledge of the distribution of aggregate withdrawals, maximizes its objective function with respect to the proportion of deposits to invest in the irreversible asset. The model in this section parallels the decision problem developed in the literature on stochastic reserve losses and the banking firm.[3] The difference is that the distribution of withdrawals in this paper is derived from consumer behavior, whereas the previous literature assumed an exogenous distribution of withdrawals. As a result, I can investigate the impact of changes in exogenous variables upon the parameters of the distribution in a systematic manner.

Section IV presents the integrated problem of the bank and N identical consumers assuming particular forms for the utility function and the distribution of an individual's income. Numerical calculations of the equilibrium for sample parameter values are presented. In particular, the increase in the social allocation of resources to investment in the irreversible asset is plotted as a function of N, the number of consumers in the economy. This serves to illustrate both the quantitative importance of deposit risk pooling and the extent of stochastic economies of scale.

Section V concludes with a discussion of this modelling strategy and considers directions for future research. The Appendix contains the proof of the form of the aggregate withdrawal function for the example used in Section IV.

II. The Representative Consumer

In this section I develop a model of the individual consumer under uncertainty in order to provide the microeconomic foundations of precautionary saving and optimal deposit-withdrawal behavior.

Before turning to the mathematical presentation of the consumer's problem it is important to discuss the assumption that the asset available to the consumer is completely irreversible and to emphasize the role that this assumption plays in a model of precautionary savings. A completely irreversible asset is one which, once committed to, cannot be undone prior to maturity, at any cost.[4] For example, if an individual invests one unit of wealth in a two-period completely irreversible asset he may obtain nothing prior to maturity and $R(= 1+r)$ in period two. In effect the principal is "locked-in" until maturity when both the principal and the yield become available.

The assumption of completely irreversible investment in this model implies that it is infinitely costly to borrow against the certain future wealth represented by the asset and/or is infinitely costly to liquidate early some portion of the rights to second-period income. This assumption is more severe than simply assuming that the ultimate borrower's physical return to investment in an income generating activity is unavailable prior to the second period.(This might result from, for example, gestation lags in setting up productive facilities, i.e., the lumber cannot be delivered until the tree grows). A secondary market could exist to permit trading in the rights to the future physical output so that when circumstances change, that which is socially irreversible could be shifted to those individuals who find it relatively most desirable to hold as a vehicle to acquire future consumption.

The assumption of complete irreversibility is extreme. It amplifies the costs associated with ex post over-investment. It represents a polar case just as the perfectly reversible investment represents another extreme. In reality there are penalties for early liquidation, broker's fees, sunk costs result from investment in asset-specific information which cannot be credibly transferred to a prospective buyer, and many other transactions costs which introduce an element of costly reversibility into the investment decision. Surely, the existence of many of the modern financial institutions is in part due to their ability to economize on these costs and reduce the cost of reversing investments. But even with these institutions some cost of reversibility remains, if only because of the scarce real resources required to supply market-making services competitively.

Whatever its quantitative significance, the qualitative significance of costly reversibility is that it provides a motive for holding precautionary balances in the absence of complete insurance markets and when 1) some uncertainty exists that is not resolved until after the portfolio decision is made, and 2) the consumer is risk averse. Where portfolio decisions are costless to undo, the consumer would invest all of his initial wealth in the interest-bearing asset.(Provided that it offered a return greater than that earned on bank deposits.) After learning of his particular income realization he would costlessly retrieve just enough of the asset to maximize two-period utility. It is the *ex ante* awareness of the *ex post* cost of reallocation in the event of a low-income realization that serves to restrain the consumer from committing all of his resources to investment and as a result generates the demand for precautionary holdings.

In the model developed in this section I will assume complete irreversibility of investment rather than specify a more elaborate environment that permits costly reversibility. The model of complete irreversibility conveys the same qualitative withdrawal-deposit behavior and has the advantage of relative simplicity. A model of penalty borrowing to facilitate *ex post* reallocation of resources is utilized in the numerical analysis of Section IV.

The individual consumer has a utility function defined over consumption in each of two periods. He is endowed with initial wealth, W, and receives a stochastic realization of income, Y, in period one. Income is distributed with density f(Y). The price of consumption goods in terms of wealth or income is constant and set to one in both periods. The consumer has two means of transferring purchasing power into the future. He may hold savings balances in a deposit account paying $R_S(= 1+r_s)$ in period two and/or invest in an irreversible asset returning $R(>R_S)$ in the second period. Deposits are assumed to be available to the consumer at any time, forfeiting only r_s if withdrawn before the second period. The investment asset is completely irreversible. Once made it is worth R in period two and zero prior to that time. The consumer's problem is to choose K, the proportion of initial wealth to commit to the irreversible asset, prior to observing the realization of stochastic income.

After the portfolio decision is made the consumer learns of his particular income realization, Y, and then allocates income and available precautionary balances, W(1-K), to current consumption, C_1, and saving (if any).[5] This second stage, or ex post, optimal allocation is made conditional on the knowledge that RWK, the income from the irreversible asset, will be available for consumption in period two.

More formally, the consumer chooses K to maximize expected utility,

(1) $$EU = \int_{-\infty}^{\infty} U(C_1, C_2) f(Y) dY$$

subject to the liquidity constraint,

(2) $C_1 \leq W(1-K) + Y$,

and the second period budget constraint

(3) $C_2 \leq RWK + R_S S$,

where

$S = W(1-K) + Y - C_1$ = savings

W = initial wealth

Y = income

$f(Y)$ = the probability density function of Y

C_1 = consumption in period 1

C_2 = consumption in period 2

K = proportion of initial wealth invested in the irreversible asset

R = return to investment, $(1+r)$

R_S = return to savings, $(1+r_s)$.

I will assume that the utility function is increasing and concave in each argument, that utility is additively separable across periods, and that there is no personal rate of time preference.

$U_1, U_2 > 0$,

$U_{11}, U_{22} < 0$,

and $U_{12} = U_{21} = 0$.[6]

The solution to this problem can be approached conveniently by separating the analysis into two stages. First one can solve for the values of C_1 and C_2 which maximize utility conditional on Y and K. Then, given these conditional maximum value functions, one can maximize expected utility with respect to K given the density of Y, f(Y). This is just a simple "backward induction step" applied to a two-period dynamic program.

Let me begin with the solution of the second stage problem. After time t = 1 the individual has chosen K and received income realization Y. He then allocates liquid balances to consumption C_1 and saving to maximize two-period utility subject to the liquidity constraint and the second period resource constraint. Mathematically one can use Lagrange's method and maximize the following expression with respect to C_1 and C_2

(4) $L = U(C_1, C_2) + \lambda[W(1-K)+Y-C_1] + \delta[RWK+R_S S - C_2]$

where λ and δ are the Lagrange multipliers associated with the liquidity constraint and the resource constraint respectively. Differentiating the above with respect to C_1, C_2, λ and δ yields

$$\frac{\partial L}{\partial C_1} = U_1(\ ) - \lambda + \delta R_S = 0$$

$$\frac{\partial L}{\partial C_2} = U_2\ (\ ) - \delta = 0$$

$\lambda \cdot [W(1-K) + Y_1 - C_1] = 0$; with complementary slackness

$\delta \cdot [RWK + R_S S - C_2] = 0$; with complementary slackness

Rearranging these expressions yields

(5) $$\frac{U_1\ (\ )}{U_2\ (\ )} = \frac{\lambda}{\delta} + R_S$$

The ratio of the marginal utilities of consumption is equal to the return to savings plus a term which is the ratio of the shadow value of liquidity in period one to the shadow value of income in period two. When, given K, Y is large enough that the consumer desires to devote some liquidity to saving for the purpose of future consumption, the liquidity constraint ceases to bind and λ becomes zero. The ratio of marginal utilities then equals the return to saving, R_S. Note also that δ, the shadow value of income in period two, is never zero because the consumer is never satiated by second-period consumption. (U_2 is greater than zero by assumption). Let us define Y* as that value of income such that in equilibrium saving is exactly equal to zero.

$$\frac{U_1(C_1)}{U_2(C_2)} = R_S$$

when $C_1 = W(1-K) + Y^*$

$$C_2 = RWK.$$

For $Y > Y^*$ the liquidity constraint will not bind and

$$\frac{U_1(C_1)}{U_2(C_2)} = R_S;$$

when,

$$C_1 = W(1-K) + Y - S,$$

$$C_2 = RWK + R_S S$$

and $S > 0$.

For $Y < Y^*$ equation (5) holds and

$$\frac{U_1(\ )}{U_2(\ )} > R_S.$$

Note also that when utility is additively separable in consumption in each period, $\frac{\partial Y^*}{\partial K}$ is increasing

$$\frac{\partial Y^*}{\partial K} = [\ 1 + \frac{U_1 U_{22}}{U_2 U_{11}} R]W > 0 \qquad (6)$$

The value Y^* separates the range of income realizations into two regions. Realizations of Y greater than Y^* constitute the region of *ex post* under-allocation to the irreversible asset. Had the consumer known the value of Y *ex ante* he could have invested more resources in the asset earning $R > R_S$ and improved his allocation. Similarly, the values of Y less than Y^* constitute the region of *ex post* over-commitment to the irreversible asset. In this region the consumer would like to transfer wealth back to period one but cannot because all of his second period wealth is "locked up" until that time.

From the analysis above one can plot C_1, C_2, and S as functions of income for a given K. This is done in Figures 2 and 3. There one can see that for low values of Y all additional income is devoted to first-period consumption. Beginning at $Y = Y^*$ some portion of increasing income will be saved for consumption in the second period. Optimal saving as a function of Y is continuous and nondifferentiable at Y^*. For values of Y less than Y^* all available liquid balances are devoted to C_1 and savings are zero. For Y greater than Y^* saving is a positive function of income. At $Y = Y^*$ saving is exactly zero.

$$S = 0 \qquad\qquad -\infty \leqq Y \leqq Y^*$$

$$(7) \qquad = S(Y) \qquad\qquad Y > Y^*$$

$$1 > \frac{dS}{dY} > 0.$$

Let us now return to the first stage problem of choosing the optimal portfolio shares to be devoted to precautionary balances and irreversible investment. Using the conditional maximum value functions for C_1, C_2, and S developed above, one can rewrite the consumer's problem as:

Maximize with respect to K

$$(8) \quad EU = \int_{-\infty}^{\infty} U(C_1(K,Y),\ C_2(K,Y))f(Y)dY,$$

$$\text{where } C_1(K,Y) = W(1-K) + Y - S$$

$$C_2(K,Y) = RWK + R_S S$$

To choose the optimal K the consumer equates, at the margin, the benefits of increased period-two consumption and increased period-one consumption when Y is high, with the costs of increasing K (due to both the increased probability of being liquidity constrained and the increased severity of that constraint) at levels of Y less than Y*.

In section IV below an example of this maximization problem will illustrate the analysis developed in this section using a particular parameterization of utility and a specific density for income. But before doing so there is one more building block to create from the above analysis, the distribution of individual deposit withdrawals.

The individual consumer initially deposits W(1-K) at the bank at time $t = 1 - \varepsilon$. After realizing income Y at $t = 1$ he can draw down any or all of that deposit to devote to consumption in period one or add to the initial deposit if income is unusually high. The net withdrawal, h, is the difference between initial deposits and savings held after income is realized.

$$(9) \quad h = W(1-K) - S$$

where S is the savings function developed above. The schedule of withdrawals as a function of income is depicted in Figure 4. For realizations of income less than Y* the consumer is liquidity constrained and withdraws all of his initial deposit. For Y > Y* some positive deposit balances are maintained and may even constitute a net deposit (a negative net withdrawal) for large realizations of Y.

In the next section I derive the distribution of aggregate withdrawals for an N consumer economy under some particular assumptions. To do that one must develop the distribution of individual net withdrawals, m(h). Recall that for all Y less than Y* the consumer withdraws all W(1-K) of his initial deposits. Thus the distribution of withdrawals will have a probability mass of magnitude F(Y*) at W(1-K), where F(Y*) is the cumulative distribution function of f(Y) evaluated Y*. The remaining portion of m(h) will be a region of density derived from a transformation of f(y).

For $Y \geq Y^*$, withdrawals are a monotonic function of income.

$$h = W(1-K) - S(Y).$$

Inverting this function and transforming variables yields

$$Y = S^{-1}(W(1-K)-h)$$

$$\frac{dY}{dh} = -S^{-1\prime}$$

$$m(h) = \frac{dY}{dh} \cdot f(S^{-1}(W(1-K)-h); \text{ for } - \infty \leq h < W(1-K).$$

The complete distribution of withdrawals is

$$(10) \quad m(h) = F(Y^*) \text{ at } h = W(1-K)$$

$$= S^{-1}f(S^{-1}(W(1-K)-h)) \text{ for } - \infty \leq h < W(1-K).$$

I will illustrate this with an example which will be used in the next two sections. I will assume that income is distributed exponentially.

$$f(Y) = \theta e^{-\theta y};\ 0 \leqq Y \leqq \infty$$

I will also derive a saving function of the form

$$S(Y) = B(y-y^*);\ \text{for } Y > Y^*$$

So $h = W(1-K) + By^* - By$

Rename $W(1-K) + By^* = \alpha$

Then $h = \alpha - By$ for $Y > Y^*$

$$\text{So} \quad y = \frac{\alpha - h}{B}$$

$$\frac{dy}{dh} = -\frac{1}{B}$$

$$\text{and } m(h) = \frac{\theta}{B} e^{(\theta/B)\,[h-\alpha]} \quad \text{for } -\infty \geqq h < W(1-K)$$

$$= F(Y^*) = 1 - e^{-\theta y^*} \text{ at } h = W(1-K).$$

This distribution is pictured in Figure 5. We can now turn to the derivation of aggregate withdrawal behavior and then to a model of the bank.

III. The Financial Intermediary

The previous section developed a model of the precautionary savings held by an individual consumer. I now turn to the systemic interaction between a community of N consumers and the financial intermediary.

Before directly analyzing the optimizing behavior of the bank one must derive the functional relationship between individual deposit-withdrawal behavior and the aggregate withdrawal behavior seen by the banking firm. Because of the difficulty of aggregation when dealing with mixed distributions I am confined to making particular assumptions about the form of the individual distribution of income. I will assume that the individual consumer's income is distributed exponentially.

$$f(Y) = \theta e^{-\theta y} \; ; \; 0 \leq Y \leq \infty$$

As was developed in the analysis at the end of the preceding section, the distribution of individual withdrawals is a transformation of the distribution of income

$$m(h) = \frac{\theta}{B} e^{\frac{-\theta [h-\alpha]}{B}} \qquad -\infty \leq h < W(1-K)$$

$$= F(Y^*) \qquad \text{at } h = W(1-K)$$

where $F(Y^*) = 1 - e^{-\theta y^*}$

From the distribution of individual withdrawals one can construct a distribution of aggregate withdrawals under the additional assumptions that all N consumers are identical and that the N incomes are independently and identically distributed. Using the theory of convolution one can then derive the distribution of the sum of N individual withdrawals, h_N, where

$$h_N = \sum_{i=1}^{N} h_i \; .$$[7]

The distribution of h_N, call it $m(h)_N$, is of the form

$$(11)\quad m(h)_N = \sum_{J=1}^{N} \frac{N!}{(N-J)!J!} [\frac{\Theta}{B} * e^{-\theta Y^*}]^J \cdot F(Y^*)^{N-J} \cdot \frac{[NW(1-K)-h]^{J-1}}{(J-1)!}$$

$$* e^{(\theta/B)[h-NW(1-K)]}$$

$$\text{for } -\infty \leqq h \leqq NW(1-K)$$

$$= F(Y^*)^N \text{ at } h = NW(1-K).$$

The proof of this is contained in the appendix.

As N, the number of independent consumers, becomes large the distribution of aggregate withdrawals approaches a normal distribution of mean $N \cdot E(h)$, and standard deviation $(NVAR(h))^{1/2}$ where

$$E(h) = W(1-K) - B/\theta e^{-\Theta Y^*}$$

$$Var(h) = e^{-\Theta Y^*} (\frac{B}{\Theta})^2 [2 - e^{-\Theta Y^*}].$$

This result is obtained by the application of the Lindberg and Levy Central Limit Theorem. In fact, the stochastic reserve loss literature originated by Edgeworth (1888) and reinvigorated by Orr and Mellon (1963) begins with the assumption that the distribution of aggregate withdrawals is exogenous, normally distributed, and centered at zero. The present analysis adds to the previous work in this area in two ways. First, when N is large enough to justify normal approximation, as it almost certainly is for any bank, the mean of the distribution is derived from individual utility maximizing behavior and is therefore a function of the parameters of the system, including the return paid on deposits, R_S. Second, an explicit form for the distribution of aggregate withdrawals permits one to examine the extent of stochastic economies of scale in banking as N increases from 1 to "large enough to use the normal approximation." I will make use of these features in section IV below.

With the aggregate withdrawal density $m(h)_N$ one can turn to the model of the financial intermediary. The financial institution collects the precautionary deposits of the N consumers in the economy and invests some portion of the proceeds in the asset returning R in period two. The intermediary's liabilities may be withdrawn without penalty, (Return = 1), at the initiative of the consumer prior to period two and will pay R_S if left in the bank until the second period. The intermediary's asset management consists of choosing K_B, the proportion of initial deposits to invest in the irreversible asset, prior to the resolution of consumer uncertainty. The remaining portion of deposits is held as reserves to meet the demand for liquidity at the end of period 1. (Timing is shown in Figure 1.)

In order to specify the intermediary's objective function it is necessary to specify the mechanism for handling those aggregate states of nature in which the total withdrawals demanded exceed the intermediary's reserves (i.e., a liquidity crisis). In what follows I will assume that the financial intermediary has access to an infinitely elastic supply of liquidity for which it must pay R_B ($\gg$R) per unit.

Alternatively one could assume that convertibility would be suspended in the event of liquidity crisis but such a convention necessitates specifying a mechanism to allocate what liquidity there is among those demanding to withdraw deposits. In addition, the possibility of being shut out at the bank's window would change the nature of the consumer's maximization problem. The precautionary balances held in a bank would no longer be "as good as the mattress."

The introduction of penalty borrowing for the banks makes it less costly for the intermediary to over-invest *ex post* in the illiquid asset than is the case for the consumer. Qualitatively, this could be interpreted as implying that the consumer has more difficult access to the short-term capital market than does a financial intermediary (perhaps due to fixed costs, reputational effects, etc.) or that a government discount window is available to the bank. In the next section where numerical estimates of the increased quantity of investment in the illiquid asset afforded by the introduction of the financial intermediary are discussed, two versions of the consumer's problem will be presented. In the first, consumers will be able to borrow at the same rate R_B as the bank. In the second consumers will not be permitted to borrow. It will then be possible to assess the relative importance of deposit pooling vs. asymmetric liquidity as forces leading to increased investment by the financial intermediary.

More formally, I will assume that the bank is a risk neutral firm maximizing expected profits with respect to K_B, the proportion of initial deposits to invest, holding R_B, R, and for the moment, R_S constant.

Mathematically, one maximizes with respect to K_B the expression

$$(12)\quad E(\pi) = (R-1)K_B\,Dep-(R_S-1)\int_{-\infty}^{Dep}[Dep-h]\,m(h)dh$$

$$-\,(R_B-1)\int_{(1-K_B)Dep}^{Dep}[h-(1-K_B)Dep]m(h)dh - F$$

where

$Dep = N * W(1-K)$ = initial deposits

$m(h)$ = distribution of aggregate withdrawals

h = aggregate withdrawals

K_B = proportion of deposits invested

R_B = penalty borrowing cost $(1+r_B)$

R_S = return to deposits held at bank $(1+r_s)$

R = return to holding the irreversible asset

F = any fixed costs incurred by the bank

There are four terms in this expression. The first represents the revenues earned from investing. The second represents the expected interest payments to depositors. The third term represents the bank's costs of going to the discount window if reserves do not cover liquidity. The fourth term, F, represents fixed costs of operation.

Differentiating the bank's objective function with respect to K_B yields:

$$(13)\quad \frac{(R-1)}{(R_B-1)} = [\,1 - M((1-K_B)Dep)],$$ [8]

where $M(h)$ = the cumulative distribution function of aggregate withdrawals. For the case developed above when incomes are distributed exponentially the C.D.F. is

$$(14)\quad M(h) = \sum_{J=1}^{N} \frac{N!}{(N-J)!J!} \cdot [\frac{\theta}{B}e^{-\theta Y*}]^J \cdot \sum_{i=1}^{J} \frac{[h-NW(1-K)]}{(J-i)!} {}^{J-i}$$

$$* (\frac{B}{\theta})^{i} * e^{\theta/B[h-NW(1-K)]}$$

$$\text{for } -\infty \leqq h \leqq NW(1-K)$$

$$= 1 \text{ for } h \geqq NW(1-K).$$

The first order condition (13) equates the marginal profit per unit of investment to the marginal expected cost to penalty financing.

Second order conditions assure that the above is a maximum.

$$(15)\quad \frac{\partial^2 E(\pi)}{\partial K_B^2} = -(R_B-1)\ Dep^2 M\ [(1-K_B)Dep] < 0$$

Taking the total differential of the equilibrium condition reveals the comparative static properties:

$$(16)\quad \frac{\partial K_B}{\partial R} = \frac{1}{(R_B-1)Dep.m(1-K_B)Dep)} > 0$$

$$\frac{\partial K_B}{\partial R_B} = \frac{-(R-1)}{(R_B-1)} \quad \frac{\partial K_B}{\partial R} < 0$$

The two extremes, $K_B = 0$ and $K_B = 1$ bear some examination because of the nature of the aggregate withdrawal distribution m(h). Since $R_B > R$, the left hand side of the equilibrium condition, (13) is less than unity. For small enough N it is possible that

$$1 > \frac{R-1}{R_B-1} > [1 - F(Y*)^N].$$

(To determine whether $K_B = 0$ or $K_B = \varepsilon$, where ε is infinitesimally greater than zero, one would have to compute the expected profits at each point.)

At the other end of the spectrum, when $K_B = 1$, if

$$1 - M(0) > \frac{R-1}{R_B-1}$$

all of the initial deposits will be invested in the irreversible asset. This scenario would develop in the case where N is large enough to justify the normal approximation and expected savings exceed W(1-K) for the representative individual. In that case the distribution of aggregate withdrawals would collapse around $N*E(h) < 0$ as N increased toward infinity. (Recall also that the mass point of magnitude $F(Y^*)^N$ goes to zero in the limit as N increases to infinity.)

Up until this point I have assumed that the bank is a monopolist and yet it treats R_S, the deposit rate, as exogenous. Alternatively, one could permit the bank to maximize profits with respect to both K_B and R_S. As the bank changed R_S it would increase the cost of deposit payments but it would also induce alterations in the consumer's portfolio behavior and therefore affect initial deposits, expected savings and the distribution of withdrawals. Another means to endogenize the deposit return R_S would be via the assumption that entry is frictionless and drives expected profits to zero. From this one could establish a schedule relating the deposit reward R_S to N the number of consumers. This would lead to a "one bank theorem" where the maximum deposit rate could be offered by one firm pooling all of the deposit risk in the economy.

In the next section of the paper I continue to assume that the deposit premium is exogenous for the purpose of a numerical example. In that exercise I will seek to measure the increased investment in the irreversible asset afforded by the introduction of a deposit-pooling financial intermediary. I will set R_S equal to one so that the consumer's portfolio allocation will be identical to what would be the case if he kept his money "in the mattress" (i.e., in the absence of a financial intermediary).

IV. Numerical Example

In this section I will develop a particular parameterization of the consumer's problem and then, for a vector of sample parameter values, compute the bank's optimal portfolio division as a function of N, the number of consumers in the economy. From these calculations I can then report the percentage increase in social investment in the irreversible asset afforded by the introduction of the deposit risk pooling financial intermediary.

To develop the particular parameterization of the consumer, I will use an additively separable two-period utility function which exhibits constant absolute risk aversion in each argument. I will also use the exponential distribution introduced in the previous section to model the distribution of consumer income. Assume that:

$$U(C_1, C_2) = -[e^{-aC_1} + e^{-aC_2}]$$

$$U_1 = ae^{-aC_1}$$

$$U_2 = ae^{-aC_2}$$

$$U_{11} = -a^2e^{-aC_1}$$

$$U_{22} = -a^2e^{-aC_2}$$

$$U_{12} = U_{21} = 0$$

and $f(Y) = \theta e^{-\theta Y} \quad 0 \leqq Y \leqq \infty$

$$E(Y) = 1/\theta$$

where a is the coefficient of absolute risk aversion and E denotes the mathematical expectation operator.

As was explained in Section II above, the consumer's problem can be separated into two stages. First, one can derive the optimal choice of C_1 and C_2 as a function of K and Y. Then these conditional maximum value functions are introduced into the objective function and the resultant expression is maximized with respect to K. For the particular specification introduced above, the conditional functions, $C_1(w,k)$ and $C_2(w,k)$ are derived by maximizing the following expression with respect to C_1 and C_2:

$$L = - [e^{-aC_1} + e^{-aC_2}] + \lambda[w(1-k) + Y - C_1] + \delta[RWK + R_S S - C_2]$$

The first order conditions imply that,

$$\frac{U_1}{U_2} = e^{-a(C_1-C_2)} = \frac{\lambda}{\delta} + R_S$$

where λ is the Lagrange multiplier associated with the liquidity constraint and δ is the Lagrange multiplier associated with the second-period resource constraint. At the point where the equilibrium savings, S, is exactly zero, the liquidity constraint ceases to bind and λ goes to zero. At such a point

(17) $C_1 = C_2 - (1/a)\ln R_S$.

At the point where S = 0 in equilibrium,

$C_1 = W(1-K) + Y^*$

$C_2 = RWK$

where Y* is the value of income at which, given K, the liquidity constraint ceases to bind. Using these two expressions for C_1 and C_2 in conjunction with equation (17) reveals that

(18) $Y^* = [(R+1)K-1]W - (1/a)\ln(R_S)$

For values of Y less than Y* savings will be zero, and all available liquidity is devoted to consumption in period one. For Y greater than Y* some positive savings will be carried over to augment the return to the irreversible asset.

$$C_1 = W(1-K) + Y - S$$

$$C_2 = RWK + R_S S$$

Substituting these expressions into (17) and solving for S reveals

$$(19) \quad S(Y) = \frac{W(1-K) + (1/a)\ln R_S - RWK}{1 + R_S} + \frac{Y}{1+R_S}$$

$$= B(Y-Y^*) \text{ for } Y \geq Y^*$$

where $B = \frac{1}{1+R_S}$ = marginal property to save out of income.

From the saving schedule it is easy to obtain the net withdrawal schedule

$$h = W(1-K) - S$$
$$= W(1-K) \quad \text{for } Y \leq Y^*$$
$$= \alpha - By \quad \text{for } Y > Y^*$$

where $\alpha = W(1-K) + BY^*$.

With the savings function and the expressions for C_1 and C_2 one can return to the consumer's portfolio decision and maximize with respect to K the expression:

$$(20) \quad EU = \int_0^\infty - [e^{-aC_1} + e^{-aC_2}]\theta e^{-\theta y} dy$$

where $C_1 = W(1-K) + Y - S$

$C_2 = RWK + R_S S$

and $S = 0 \quad 0 \leq Y \leq Y^*$

$$= \frac{1}{1+R_S}[Y-Y^*] \quad Y \geq Y^*$$

Differentiating this expression with respect to K, making use of Leibnitz' Rule, and setting the resultant expression equal to zero yields

$$(21) \quad \frac{\theta^* R_S^*}{a+\theta}[1 - e^{(a+\theta)Y^*}] + R[e^{\theta Y^*} - 1] + \frac{RS^{\frac{2R_S}{1+R_S}} \cdot \theta \cdot (R-R_S)}{\frac{a^* R_S}{1+R_S} + \theta} = 0$$

To insure that the K which solves this condition is a maximum the second-order condition must be negative at the optimum

$$\frac{\partial^2 EU}{\partial K^2} = \theta(R+1)We^{\theta Y^*}[R - R_S e^{aY^*}]$$

This expression is negative provided that

$$(22) \quad K^* > \frac{1}{1+R} + \frac{\ln R}{aW(1+R)}$$

where K* denotes the value of K that satisfied the first-order condition.

An additional point is worth raising at this time. To insure that Y* is within the range of Y, it must be greater than zero. This in turn implies that:

$$(23) \quad K^* > \frac{1}{1+R} + \frac{\ln R_S}{aW(1+R)}$$

Because $R > R_S$ by assumption, condition (23) is satisfied provided that the K which satisfied the first order conditions is a maximum, condition (22) above. The satisfaction of this criterion assures one that there is some ex ante region of Y in which the consumer will experience liquidity constraints. The upper limit on K is one, so Y* is always less than or equal to $(RW-1/a\ln R_S)$. This insures that one portion of the range of $[0, \infty]$ represents ex post under-commitment to the irreversible asset.

The equilibrium condition above is a trifle cumbersome and in general numerical computation will be needed to solve for K*. There is, however, one special case that permits an analytic solution for K*. This case exists when $\theta = a$. There is no economically compelling reason why the degree of absolute risk aversion should be equal to the reciprocal of expected income, but in such a case one can solve for K* using the quadratic formula and obtain

$$K^* = \frac{1}{1+R} + \frac{\ln(R+(R^2-2R_S Z)^{1/2})}{aW(1+R)}$$

where $Z = R - \frac{R_S}{2} - \frac{R_S \frac{2R_S}{1+R_S} \cdot [R-R_S]}{1 + \frac{R_S}{1+R_S}}$

Comparative static properties are generally ambiguous, but if one assumes that interest on the irreversible asset is less than 100 percent ($R < 2$) and that expected income and initial wealth are of approximately the same size (i.e., $\theta w \sim 1$), one can evaluate K^* when $R_S = 1$ and obtain,

$$\left.\frac{dK^*}{dR_S}\right|_{R_S=1} < 0$$

$$\left.\frac{dK^*}{dR}\right|_{R_S=1} > 0$$

$$\left.\frac{dK^*}{dE(Y)}\right|_{R_S=1} = \frac{\ln(R + (R^2-2/3R-1/3)^{1/2})}{W(1+R)} > 0 \qquad \text{(for } R > 1\text{)}$$

$$\left.\frac{dK^*}{dW}\right|_{R_S=1} = \frac{-\ln(R+(R^2-2/3R-1/3)^{1/2})}{W(1+R)} < 0 \qquad \text{(for } R > 1\text{)}$$

For the extremely special case just derived the comparative statics imply that: raising the deposit rate will increase savings deposits (K^* falls); increasing the return to irreversible investment attracts more investment; increasing expected income, and therefore expected liquidity, will increase investment; and finally increasing the consumer's initial endowment, W, decreases the proportion of wealth devoted to irreversible investment. Also note that at $R_S = 1$, K^* is a maximum for R greater than one. Referring to (22) above

$$K^*\Big|_{R_S=1} = 1/1+R + \frac{\ln(R+(R^2-(2/3)R-1/3)^{1/2})}{aW(1+R)} > \frac{1}{1+R} + \frac{\ln R}{aW(1+R)}$$

Let me now use the consumer model developed in this section in conjunction with the model of the bank developed in the previous section and a vector of sample parameter values to compute numerically the increase in social investment in irreversible opportunities afforded by the introduction of a deposit risk pooling financial intermediary. Conceptually I will be analyzing two economies, both consisting of N identical consumers with independently distributed incomes, but in one economy I will introduce a financial intermediary who collects deposits from the N consumers and invests in the irreversible asset. One can then compare the two economies to assess the increment in irreversible investment attributable to the presence of indirect finance.

In what follows I will assume that the environment is characterized by the following parameter values:

$R = 1.10$; return to investment in period two

$R_S = 1.0$; return to saving in period two.

$R_B = 1.25$; penalty rate

$W = 2$; initial wealth

$\theta = .5$; reciprocal of expected income

$a = .5$; the degree of absolute risk aversion

$F = 0$; fixed cost of banking operation.

In this section I will assume that the supply of irreversible investment assets is infinitely elastic at $R = 1.10$. Under this assumption the efficiencies resulting from the introduction of a financial intermediary will be reflected solely in the increased quantity of irreversible investment and not at all by changes in the price of credit to the ultimate borrower.

I also assume, as is implied in this specification, that the deposit return, R_S, is treated parametrically rather than as a variable determined by the intermediary. With $R_S = 1.0$ the consumer is presented with the same environment in the presence of the financial intermediary that exists in its absence. I further assume that when indifferent between the deposit account and holding money "in the mattress," the consumer will choose to put his money in the bank. As mentioned above, the introduction of free entry would serve to endogenize the deposit rate. If the consumer were to respond to positive interest on deposits by increasing liquid balances, ($dK/dR_S < 0$) then the introduction of the competitive financial institution would increase the proportion of total investment done indirectly when compared with the case displayed below. Furthermore, in utility terms the existence of positive deposit interest will make the consumer no worse off and possibly better off. (Revealed Preference).

In Table 1 and Figure 5 below, I report the results of two experiments. In the first instance, consumers are unable to borrow, R_B equals ∞ , whereas banks may borrow at $R_B = 1.25$. The implication of this is that the cost of reversing investment is not the same for the two types of market participants, and therefore banks would be likely to undertake a larger share of total investment than would be the case if both participants were faced with the same cost of reversibility.

In order to filter out this effect of asymmetric liquidity a second experiment is undertaken by slightly modifying the consumer model presented above to permit consumers to borrow at R_B if they so desire.[9]

The purpose of this exercise is to isolate the effects of deposit risk pooling, and this modification serves that purpose and does not alter the consumer problem presented above in any fundamental way. In fact, for the parameter values listed above, K*, the consumer's optimal share of initial wealth devoted to the irreversible asset changes from .663 to .706 and R_B moves from ∞ down to 1.25.

Before turning to the numerical presentation I must mention one other asymmetry between the bank and consumers. Consumers in this model are risk averse, whereas the bank is risk neutral. This asymmetry tends to amplify the extent to which the risk pooling depository institution increases the social commitment to irreversible investment. While I do not make any attempt to remove this influence here, I can present a baseline measure of its importance by examining the case where N=1 and both the consumer and the bank have access to liquidity at the penalty rate. In such a case, both asymmetric liquidity and deposit risk pooling cease to be a factor in promoting investment in the irreversible asset. The consumer in the absence of the financial intermediary will invest NWK* directly. After introducing the bank, the quantity of investment increases to $NWK + NW(1-K)K_B$. The percentage increase in irreversible investment afforded by the introduction of the financial intermediary is therefore;

$$(24) \quad \Delta I = \frac{(1-K^*)K_B}{K^*} \text{ . }^{10}$$

When N=1, the data in the second column of Table 1 reveal that ΔI = 1.7%. This is very small when compared with the magnitude of ΔI as N gets large. It appears that in relation to the economies of risk pooling, the contribution of asymmetric risk aversion to the increase in investment attributable to the bank is miniscule.

Let me now turn to the examination of Table 1 and its graphical counterpart in Figure 5. The result plotted there is the relation between ΔI, the percentage increase in social investment afforded by the introduction of the deposit pooling intermediary, and N, the number of consumers holding precautionary balances at the bank. Two relations are plotted in Figure 5. The upper one corresponds to the case where consumers have equal access to emergency liquidity. As one would expect, when banks have cheaper access to funds (i.e., less costly reversibility) than do consumers, the proportionate increase in investment resulting from the introduction of the bank is greater.

Perhaps more interesting is the trajectory of ΔI as N increases. The results indicate that the efficiencies introduced by deposit risk pooling are quite substantial as N increases from one. But the marginal increment in ΔI becomes quite small as N increases beyond 30. This analysis leads one to the conclusion that the increased allocative efficiency attributable to the existence of the financial intermediary's pooling of deposits is significant, but, given that banks have hundreds if not thousands of customers, the marginal stochastic economies of scale are so small that the determination of optimal firm size is probably dominated by other considerations.

The calculation displayed in this section represents a rather crude attempt to investigate the nature of the economies attributable to the presence of a risk pooling financial intermediary. While it is the author's hope that this exercise provides the reader with a qualitative feel for the forces at work in this model economy, the results presented here are a special case and must be viewed cautiously. A more sophisticated analysis would, as described above, endogenize the deposit rate R_S and perhaps the asset return R. The author is currently developing a model with free entry in the financial sector and an asset supply schedule with finite elasticity. I hope to include presentation of that model in subsequent versions of this paper.

V. Conclusion

In the previous sections I have sought to construct a model of the interaction between consumers, their holding of precautionary balances, a financial intermediary, and investment in illiquid assets. I have not offered a theory of the illiquidity/irreversibility of assets but have merely assumed that there exists some sunk cost to be associated with investment. By making extreme assumptions about the nature of the illiquidity it was possible to develop a model of consumer deposits based upon individual optimizing behavior. It was then possible to derive the distribution of the aggregate deposit pool held by the bank if one assumed that incomes were independently distributed across individuals. Given the distribution of aggregate deposits, an expected profit maximizing bank could then choose to what extent it could invest in the illiquid asset. A particular parameterization of the problem was solved numerically to illustrate the significance of the role of the financial intermediary.

Obviously, the quantitative importance of this aspect of financial intermediation depends upon the degree of friction in the financial markets. Perhaps this is a better model of the nineteenth century bank, when secondary markets were less efficient (or nonexistent), than it is of banks in 1983.

In future research several extensions and modifications would be desirable. First and foremost, the interest rates R and R_S can be made endogenous as mentioned above in sections III and IV. This will permit more satisfactory measurement of the efficiencies gained by deposit risk pooling. On a slightly larger scale, it would be interesting to model this process in an overlapping generations context to permit analysis of intertemporal as well as contemporaneous risk pooling. It would also be desirable to imbed this model of financial allocation into a general equilibrium model and to examine the implications of monetary policy and multiple deposit expansion. Finally, and perhaps most importantly, it would be nice to dispense with the assumption of complete independence of stochastic income across consumers in order to model the impact of aggregate real fluctuations upon the ability of the bank to provide transformation services and vice versa.

FOOTNOTES

*The author is a staff economist in the International Finance Division. This paper represents the views of the author and should not be interpreted as reflecting those of the Board of Governors of the Federal Reserve System or other members of its staff. I would like to thank Alan Blinder, William Branson, Stephen Goldfeld, and Dwight Jaffee for their comments and suggestions. I would also like to thank Ruby Brooks for the preparation of the manuscript.

[1] The survey of Baltensperger (1980) is very useful in organizing the material in this literature.

[2] This point, and the subsequent developments explaining the absence of complete insurance by relying on asymmetric information, were first made explicit by Diamond and Dybvig (1983).

[3] This literature began with Edgeworth (1888). More recent contributions include Orr and Mellon (1961) and Ratti (1979). The Baltensperger survey mentioned in footnote 1 also concisely summarizes the work in this area.

[4] For a discussion of irreversibility in assets see the Tobin manuscript, Chapter 2. For a model of irreversibility which is much more sophisticated than is needed for the purposes of this paper see Bernanke (1983). In that paper the author explores the concept of "option value" when applied to durable irreversible capital investment.

[5] See Figure 1.

[6] Subcripts of utility denote partial derivatives.

$$(\text{i.e., } U_i = \frac{\partial U}{\partial C_i}.)$$

[7]Drake (1967) has an excellent discussion of convolution theory.

[8]This condition determines K_B. The bank then invests $NW(1-K)K_B$ in the irreversible asset. It is this quantity that measures the social saving on precautionary holdings due to deposit risk pooling.

[9]The introduction of consumer borrowing will divide the second stage optimization into three regions. For low values of Y the consumer will borrow. In an intermediate range he will neither borrow nor lend. For Y greater than Y* he will save. The region of borrowing is defined by Y less than Y_B, where Y_B is defined by

$$\frac{U_1(C_1)}{U_2(C_2)} = R_B$$

when

$$C_1 = W(1-K) + Y_B$$

$$C_2 = RWK.$$

For Y between Y_B and Y*, the consumer neither borrows nor lends. This range is defined by

$$R_S \leq \frac{U_1(C_1)}{U_2(C_2)} \leq R_N$$

when

$$C_1 = W(1-K) + Y$$

$$C_2 = RWK.$$

For Y greater than Y* the analysis is as presented in the text.

For the particular example used in the text augmented by borrowing, at R_B, the first order condition evaluated at $R_S = 1$ becomes:

$$K^* = \frac{1}{R+1} + \cdot \frac{1}{W(R+1)\alpha} \cdot \ln\left[1 - \frac{K_1+K_2}{K_3}\right]$$

$$\text{where } \alpha = \frac{aR_B}{1+R_B} + \theta$$

$$K_1 = R.[R_B{}^{\theta/a} - 1] + \frac{\theta}{a+\theta}\left[1 - R_B \frac{a+\theta}{a}\right]$$

$$K_2 = \frac{(R-1)\theta}{a/2+\theta}$$

$$K_3 = \frac{\theta[R-R_B]}{R_B} \cdot R_B{}^{1/(1+R_B)} * \frac{1}{\alpha}$$

[10] The percentage increase due to the introduction of the financial intermediary is,

$$\frac{NWK + NW(1-K)K_B - NWK}{NWK} = \frac{(1-K)K_B}{K}$$

Figure 1

Timing

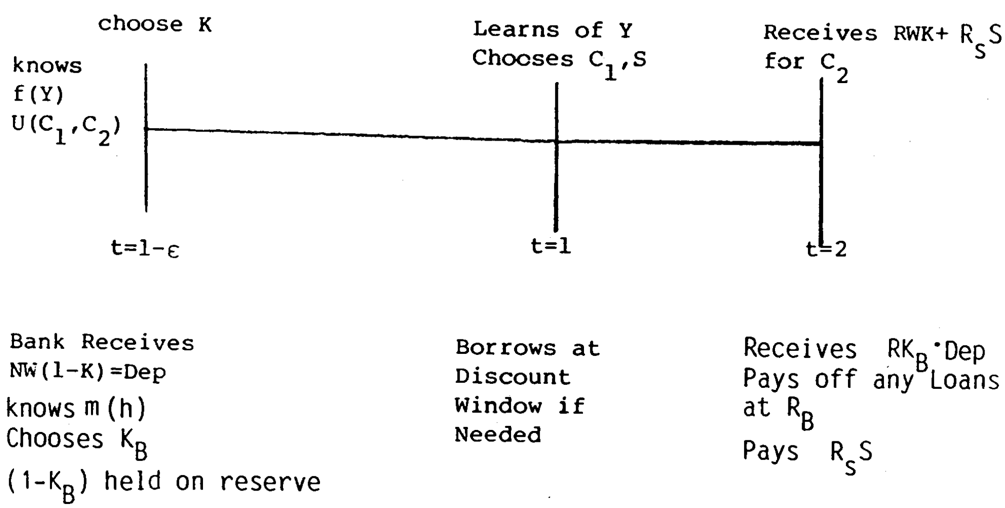


Figure 2

Consumption

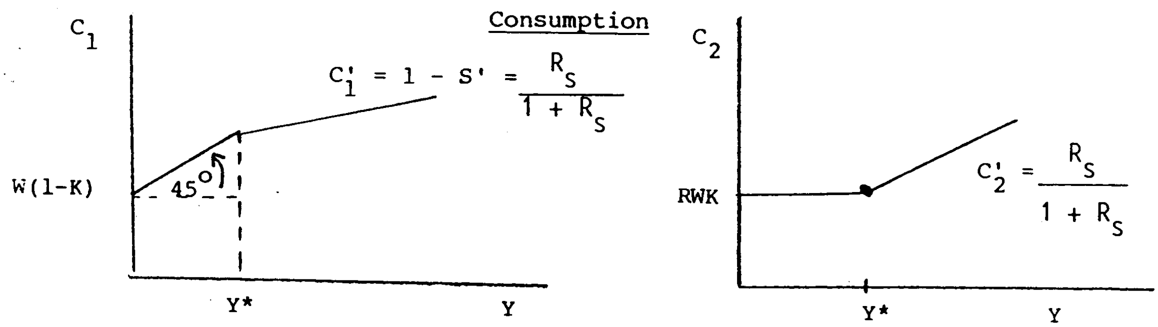


Figure 3

Saving and Deposit Withdrawals

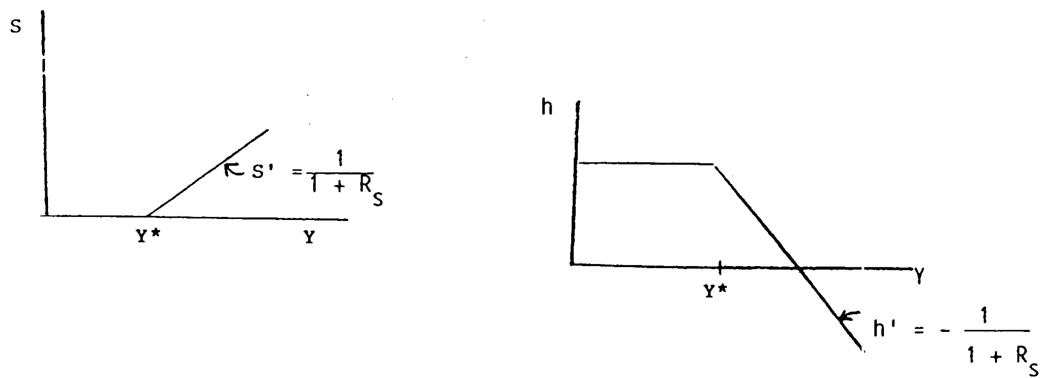
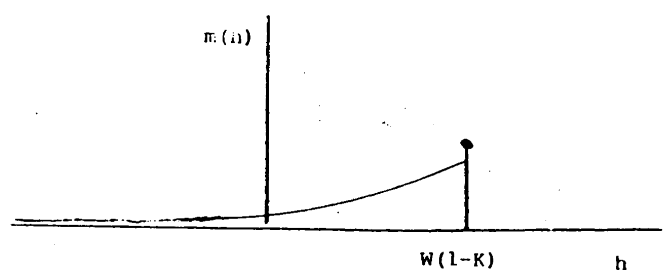


Figure 4

P.D.F. of Deposit Withdrawals



Mass Point at W(1-K) of Magnitude F(Y*)

Density $\frac{\theta}{B} e^{\theta/B\left[h-\alpha\right]}$ $-\infty \leq h \leq W(1-K)$

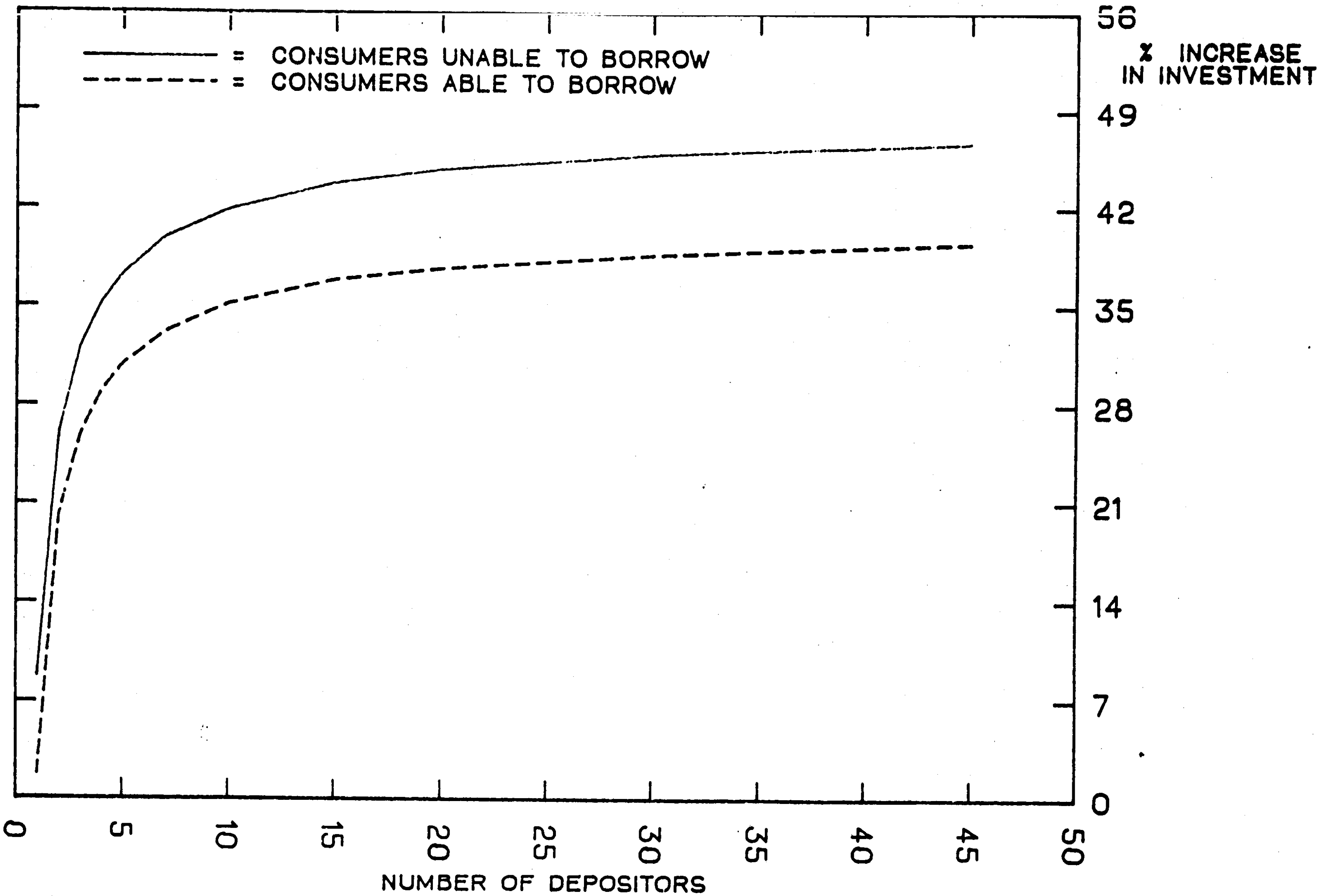
TABLE 1

The Percentage Increase in Investment

Due to Deposit Risk Pooling

# of Consumers	ΔI No Consumer Borrowing	ΔI With Consumer Borrowing
1	8.7	1.7
2	26.0	20.1
3	32.1	25.9
4	35.3	29.0
5	37.3	30.9
7	39.9	33.2
10	41.9	35.2
15	43.8	36.9
20	44.8	37.7
30	45.9	38.7
45	46.7	39.5

THE % INCREASE IN INVESTMENT DUE TO DEPOSIT RISK POOLING
FIGURE 5



APPENDIX

Proof by Induction of the Form of the Density of N Truncated Exponentials

$$m(h)_N = \sum_{J=1}^{N} \frac{N!}{J!(N-J)!} \cdot \left[\frac{\theta}{B} e^{-\frac{\alpha\theta}{B}}\right]^J \cdot F(Y^*)^{N-J} \cdot \frac{\left[NW(1-K)-h\right]^{J-1}}{(J-1)!} \cdot e^{\frac{\theta}{B}\left[h-(N-J)W(1-K)\right]} \quad \text{for } -\infty \leq h < NW(1-K)$$

$$= F(Y^*)^N \quad \text{at} \quad NW(1-K)$$

$$m(h)_1 = \frac{\theta}{B} e^{-\alpha\theta/B} \cdot e^{h\theta/B} \qquad -\infty \leq h < W(1-K)$$

$$= F(Y^*) \quad \text{at} \quad W(1-K) \quad \text{at} \quad h = W(1-K)$$

$$\text{where} \quad F(Y^*) = \int_0^{Y^*} \theta e^{-\theta Y} dY = \left[1 - e^{-\theta Y^*}\right]$$

I wish to show that the convolution of $M(h)_N$ and $M(h)_1$ equals $M(h)_{N+1}$.

Each density is comprised of an impulse and a region of continuous density. Thus there will be four components in the convolution.

(1) $F(Y^*)^{N+1}$ impulse at $(N+1)W(1-K)$ (2 impulses)

(2) $F(Y^*)^N \frac{\theta}{B} e^{\frac{\left[h-nW(1-K)-\alpha\right]}{B}\theta}$ (1 impulse & 1 density)

(3) $$\sum_{J=1}^{N} \frac{N!}{J!(N-J)!} \cdot \left[\frac{\theta}{B} e^{-\frac{\theta\alpha}{B}}\right]^J F(Y^*)^{N+1-J} \cdot \frac{\left[(N+1)W(1-K)-h\right]^{J-1}}{(J-1)!} \cdot e^{\frac{\left[h-(N+1-J)W(1-K)\right]}{B}}$$

$$(4) = \int_{N+1-nW(1-K)}^{nW(1-K)} m(h)_1 \, m\,(h_{N+1}-h_1)_N \, dh_1 \qquad h_{N+1} = h_N + h_1$$

$$= \sum_{J=1}^{N} \frac{N!}{J!(N-J)!} \cdot \left[\frac{\theta}{B} e^{-\alpha\theta/B}\right]^{J+1} \cdot F(Y^*)^{N-J} \, e^{\theta/B\left[h_{N+1}-(N-J)W(1-K)\right]} \cdot \int_{h_{N+1}-NW(1-K)}^{W(1-K)} \frac{\left[NW(1-K)-h_{N+1}+h_1\right]^{J-1}}{(J-1)!}$$

$$(4) = \sum_{J=1}^{N} \frac{N!}{J!(N-J)!} \cdot \left[\frac{\theta}{B} e^{-\frac{\alpha\theta}{B}}\right]^{J+1} \cdot F(Y^*)^{N-J} \cdot e^{\frac{\theta\left[h-(N-J)W(1-K)\right]}{B}} \frac{\left[(N+1)W(1-K)-h\right]^{J}}{J!}$$

Substitution of index $d = J+1$ into (4) yields

$$\sum_{d=2}^{N+1} \frac{N!}{(d-1)!(N+1-d)!} \cdot \left[\frac{\theta}{B} e^{-\frac{\alpha\theta}{B}}\right]^{d} F(Y^*)^{(N+1-d)} \cdot e^{\theta/B\left[h-(N+1-d)W(1-K)\right]} \cdot \frac{\left[(N+1)W(1-K)-h\right]^{d-1}}{(d-1)!}$$

Also note that if $d = 1$, the above expression would yield

$$\frac{\theta}{B} e^{-\frac{\alpha\theta}{B}} F(Y^*)^{N} e^{\theta/B\left[h-NW(1-K)\right]}$$

which is identical to (2) above.

One can then express the convolution as

$$\mathfrak{m}(h)_{n+1} = F(Y^*)^{N+1} \quad \text{at} \quad (N+1)W(1-K)$$

$$\sum_{d=1}^{N+1} \frac{N!}{(d-1)!\,(N+1-d)!} \cdot \left[\frac{\theta}{B} e^{-\frac{\alpha\theta}{B}}\right]^{d} F(Y^*)^{N+1-d} \cdot e^{\theta/B\left[h-\left[n+1-d\right]W(1-K)\right]} \cdot \frac{\left[(N+1)W(1-K)-h\right]^{J-1}}{(d-1)!}$$

$$+ \sum_{J=1}^{N} \frac{N!}{J!\,(N-J)!} \cdot \left[\frac{\theta}{B} e^{-\frac{\alpha\theta}{B}}\right]^{J} F^*(Y)^{N+1-J} \cdot e^{\theta/B\left[h-(n+1-J)W(1-K)\right]} \frac{\left[(N+1)W(1-K)-h\right]^{J-1}}{(J-1)!}$$

$$-\infty \leq h < (N+1)W(1-K)$$

The two terms involving summation can then be combined over the range $J = 1, N$ to yield

$$\mathfrak{m}(h)_{N+1} = F(Y^*)^{N+1} \quad \text{at} \quad (N+1)W(1-K)$$

$$= \sum_{J=1}^{N} \left[\frac{N!}{J!\,(N-J)!} + \frac{N!}{(J-1)!\,(N+1-J)!}\right] \cdot \left[\frac{\theta}{B} e^{-\frac{\alpha\theta}{B}}\right]^{J} F(Y^*)^{N+1-J} \cdot e^{\frac{\theta}{B}\left[h-(n+1-J)W(1-K)\right]} \cdot \frac{\left[(N+1)W(1-K)-h\right]}{(J-1)!}^{J-1}$$

Plus the term from the first of the two summations evaluated at $d = N + 1$

$$(^{**}) = \left[\frac{\theta}{B} e^{-\frac{\alpha\theta}{B}}\right]^{N+1} \cdot e^{\theta/B\left[h\right]} \cdot \frac{\left[(N+1)W(1-K)-h\right]^{N}}{N!}$$

And $\dfrac{N!}{J!\,(N-J)!} + \dfrac{N!}{(J-1)!\,(N+1-J)!}$

$$= \frac{N!\,(N+1-J)}{J!^*(N+1-J)!} + \frac{N!\,J}{J!\,(N+1-J)!}$$

$$= \frac{(N+1)!}{J!\,(N+1-J)!}$$

So the summation terms become

$$\sum_{J=1}^{N} \frac{(N+1)!}{J!\left[(N+1)-J!\right]} \cdot \left[\frac{\theta}{B} e^{-\frac{\theta\alpha}{B}}\right]^{J} \cdot F(Y^*)^{N+1-J} \, e^{\theta/B\left[h-(N+1-J)W(1-K)\right]} \cdot \frac{\left[(N+1)W(1-K)-h\right]^{J-1}}{(J-1)!}$$

were thus evaluated at $J = N + 1$ the result would be

$$\left[\frac{\theta}{B} e^{-\frac{\theta\alpha}{B}}\right]^{N+1} \cdot e^{\theta/Bh} \cdot \frac{\left[(N+1)W(1-K)-h\right]^{N}}{N!}$$

Which is identical to the term (**) above.

One can then extend the upper limit on the summation to $N + 1$ and express the convolution as

$$m\,(h)_{N+1} = F(Y^*)^{N+1} \text{ at } (N+1)W(1-K)$$

$$= \sum_{J=1}^{N+1} \frac{(N+1)!}{J!(N+1-J)!} \cdot \left[\frac{\theta}{B} e^{-\frac{\theta\alpha}{B}}\right]^{J} \cdot F(Y^*)^{N+1-J} \cdot e^{\theta/B\left[h-N+1-J\right]W(1-K)]} \cdot \frac{\left[(N+1)W(1-K)-h\right]^{J-1}}{(J-1)!}$$

for $-\infty \leq h < (N+1)W(1-K)$.

References

Baltensperger, E., 1980, "Alternative Approaches to the Theory of the Banking Firm," Journal of Monetary Economics 6, pp. 1-37.

________________, 1972, "Economies of Scale, Firm Size and Concentration in Banking," Journal of Money, Credit, and Banking 4, pp. 467-489.

Benston, G. J. and C. W. Smith, 1976, "A Transaction Cost Approach to the Theory of Financial Intermediation," Journal of Finance 31, No. 2, pp. 215-32.

Bernanke, B. S., 1983, "Irreversibility, Uncertainty and Cyclical Investment," Quarterly Journal of Economics #1, pp. 85-106.

Diamond, D. and P. H. Dybvig, 1983, "Bank Runs, Deposit Insurance, and Liquidity," Journal of Political Economy 91, No. 3, pp. 401-419.

Drake, A. W., Fundamentals of Applied Probability Theory, New York, McGraw-Hill Co., 1967.

Edgeworth, F. Y., 1888, "The Mathematical Theory of Banking," Journal of the Royal Statistical Society, v. 51, Part 1, pp. 113-127.

Lavington, F., The English Capital Market, London Methven & Co., Ltd., 1921.

Orr, D. and W. G. Mellon, 1961, "Stochastic Reserve Losses and Expansion of Bank Credit," American Economic Review 51, pp. 614-23.

Ratti , R. A., 1979, "Stochastic Reserve Losses and Bank Credit Expansion," Journal of Monetary Economics 5, pp. 283-94.

Tobin, J., Unpublished Money Manuscript.

International Finance Discussion Papers

IFDP NUMBER	TITLES	AUTHOR(s)
	1986	
289	Incomplete Insurance, Irreversible Investment, and the Microfoundations of Financial Intermediation	Robert A. Johnson
288	The Yen-Dollar Relationship: A Recent Historical Perspective	Manuel H. Johnson Bonnie E. Loopesko
287	Should Fixed Coefficients be Reestimated Every Period for Extrapolation?	P.A.V.B. Swamy Garry J. Schinasi
286	An Empirical Analysis of Policy Coordination in the U.S., Japan and Europe	Hali J. Edison Ralph Tryon
285	Comovements in Aggregate and Relative Prices: Some Evidence on Neutrality	B. Dianne Pauls
284	Labor Market Rigidities and Unemployment: the Case of Severance Costs	Michael K. Gavin
283	A Framework for Analyzing the Process of Financial Innovation	Allen B. Frankel Catherine L. Mann
282	Is the ECU an Optimal Currency Basket?	Hali J. Edison
281	Are Foreign Exchange Forecasts Rational? New Evidence from Survey Data	Kathryn M. Dominguez
280	Taxation of Capital Gains on Foreign Exchange Transactions and the Non-neutrality of Changes in Anticipated Inflation	Garry J. Schinasi
279	The Prospect of a Depreciating Dollar and Possible Tension Inside the EMS	Jacques Melitz
278	The Stock Market and Exchange Rate Dynamics	Michael K. Gavin
277	Can Debtor Countries Service Their Debts? Income and Price Elasticities for Exports of Developing Countries	Jaime Marquez Caryl McNeilly
276	Post-simulation Analysis of Monte Carlo Experiments: Interpreting Pesaran's (1974) Study of Non-nested Hypothesis Test Statistics	Neil R. Ericsson

Please address requests for copies to International Finance Discussion Papers, Division of International Finance, Stop 24, Board of Governors of the Federal Reserve System, Washington, D.C. 20551.

International Finance Discussion Papers

IFDP NUMBER	TITLES	AUTHOR(s)
275	A Method for Solving Systems of First Order Linear Homogeneous Differential Equations When the Elements of the Forcing Vector are Modelled as Step Functions	Robert A. Johnson
274	International Comparisons of Fiscal Policy: The OECD and the IMF Measures of Fiscal Impulse	Garry Schinasi
273	An Analysis of the Welfare Implications of Alternative Exchange Rate Regimes: An Intertemporal Model with an Application	Andrew Feltenstein David Lebow Anne Sibert
	1985 (partial listing)	
272	Expected Fiscal Policy and the Recession of 1982	William H. Branson Arminio Fraga Robert A. Johnson
271	Elections and Macroeconomic Policy Cycles	Kenneth Rogoff Anne Sibert
270	Assertion Without Empirical Basis: An Econometric Appraisal of *Monetary Trends in ... the United Kingdom* by Milton Friedman and Anna J. Schwartz	David F. Hendry Neil R. Ericsson
269	Canadian Financial Markets: The Government's Proposal for Reform	Garry J. Schinasi
268	Was It Real? The Exchange Rate Interest Differential Relation, 1973-1984	Richard Meese Kenneth Rogoff
267	The U.K. Sector of the Federal Reserve's Multicountry Model: The Effects of Monetary and Fiscal Policies	Hali J. Edison
266	Optimal Currency Basket in a World of Generalized Floating: An Application to the Nordic Countries	Hali J. Edison Erling Vardal
265	Money Demand in Open Economies: A Currency Substitution Model for Venezuela	Jaime Marquez
264	Comparing Costs of Note Insurance Facilities and Eurocredits	Rodney H. Mills
263	Some Implications of the President's Tax Proposals for U.S. Banks with Claims on Developing Countries	Allen B. Frankel