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EXACT AND APPROXIMATE MULTI-PERIOD MEAN-SQUARE FORECAST ERRORS FOR DYNAMIC ECONOMETRIC MODELS

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ABSTRACT

Both future disturbances and estimated coefficients contribute to the uncertainty in model-based *ex ante* forecasts, but only the first source is usually taken into account when calculating confidence intervals for practical applications. Schmidt (1974) and Baillie (1979) provide an easily computable second-order approximation to the mean-square forecast error (MSFE) for linear dynamic systems which recognizes both sources of uncertainty. To assess the accuracy of their approximation, and thus its usefulness, we compare it with three sets of estimates of the *exact* MSFE for the univariate AR(1) process: Monte Carlo estimates for OLS, analytically based values for OLS, and Monte Carlo estimates for maximum likelihood. We find that the Schmidt-Baillie formula is a good approximation to the exact MSFE, and that it helps explain why the exact MSFE can *decrease* as the forecast horizon increases. In fact, for dynamics typical to econometric models, the MSFE often has a *maximum* at a forecast horizon of one to twelve periods, i.e., at horizons that are of principal concern to forecasters and policy makers.

Key words and phrases: approximations, autoregressive models, confidence intervals, dynamics, forecasts, maximum likelihood, mean-square forecast error, Monte Carlo, statistical inference, time series.

Exact and Approximate Multi-period Mean-square Forecast Errors for Dynamic Econometric Models

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1. Introduction

In practice, numerous factors contribute to the uncertainty associated with model-based forecasts, including the inherently stochastic nature of the process generating the data and the imprecision of coefficient estimates.[2] Confidence bands for forecasts, if computed, typically take account of the first source of uncertainty, but not the second. In an extensive Monte Carlo study of the univariate AR(1) process, Orcutt and Winokur (1969) obtain unbiased estimates of the *exact* least-squares based mean-square forecast error (MSFE) which accounts for both these sources of uncertainty. Hoque, Magnus, and Pesaran (1988) (hereafter, HMP) derive an analytical expression for the exact MSFE for the AR(1) process. These two papers show that coefficient uncertainty can substantially increase the MSFE over and above the contribution from the inherent uncertainty.

Although both analyses are significant steps in properly interpreting forecasts, they have important limitations. First, as HMP note, numerical evaluation of their exact formula is computationally burdensome for any forecast horizons but very short ones (e.g., for more than four periods ahead). Given the availability of high-frequency data, longer horizons often are of interest in economic, business, and policy applications. Second, their formula is restricted to the univariate first-order process, and exact generalizations to multivariate multiple-lag econometric systems seem unlikely. Third, although Monte Carlo

[1]The authors are staff economists in the Division of International Finance. The views expressed in this paper are solely the responsibility of the authors and should not be interpreted as reflecting those of the Board of Governors of the Federal Reserve System or other members of its staff. Helpful discussions with and comments from Julia Campos, Dale Henderson, David Hendry, David Howard, Jan Magnus, Doug McManus, and Ted Truman are gratefully acknowledged. We are indebted to Ned Prescott for assistance in preparing the figures.

[2]Additional sources of uncertainty include the choice of model specification and errors in data measurement. As important as they are, those sources are beyond the scope of our paper, so we ignore them.

methods permit estimating the exact MSFE for more general models, such estimates are subject to the *imprecision* inherent in Monte Carlo simulation and the *specificity* of choosing a given model and set of coefficients rather than some other. A need exists for a formula for the MSFE which can be implemented and computed with ease for any linear dynamic system at any forecast horizon and which accounts for both inherent and coefficient uncertainty.

Schmidt (1974) and Baillie (1979) provide a solution via a simple approximation to the exact MSFE. Our paper ascertains the accuracy of their approximation by comparing it with the results from Orcutt and Winokur and HMP, and with a further study conducted herein on the MSFE of the maximum likelihood estimator for the stationary AR(1) model. We find that their approximation is remarkably accurate over a wide range of sample sizes, parameter values, and forecast horizons, giving support for its use in empirical practice.

Section 2 briefly reviews the derivation of the Schmidt-Baillie approximation and discusses its analytical properties. In particular, the approximation's formulation provides an intuitive explanation of why the MSFE can *decrease* as the forecast horizon increases, behavior which Hoque, Magnus, and Pesaran find surprising. Section 3 shows that the deviations between the approximation and the exact MSFE for the AR(1) model are numerically small for most practical purposes, except in two cases: small samples with extreme values of the autoregressive coefficient, and forecast horizons approaching the distance at which the exact MSFE is infinite. To assess the sensitivity of these findings, Section 4 compares the Schmidt-Baillie approximation with Monte Carlo estimates of the exact MSFE using an alternative asymptotically equivalent estimator, maximum likelihood: the approximation is quite accurate, both for short forecast horizons and for longer horizons at which the MSFE for OLS is infinite. As justification for evaluating the exact MSFE for maximum likelihood at such horizons, we show that any truncated estimator has a finite exact MSFE at *all* forecast horizons, and that, for some truncated estimators such as maximum likelihood, the exact MSFE is bounded, regardless of the forecast horizon. That identifies how sensitive the condition for the existence of the MSFE for OLS is to minor changes in distributional assumptions because (e.g.) a truncated OLS estimator might be truncated at only very large values (which occur very infrequently) and yet would have a finite MSFE at all forecast horizons.

2. An Approximation to the Multi-period Mean-Square Forecast Error

Schmidt (1974) and Baillie (1979) provide an approximation to the MSFE, albeit in two distinct contexts. Schmidt approximates the MSFE for the linear dynamic simultaneous equations model where a subset of the variables are strongly exogenous and are known for the forecast period. Baillie's framework allows for such strongly exogenous variables but, if they are present, requires that they be forecast as well (i.e., true *ex ante* forecasting). Our exposition follows Chong and Hendry (1986) because of the the latter's accessibility. To help in understanding the properties of the *exact* MSFE, this section sketches the derivation of the Schmidt-Baillie approximation and discusses its analytical properties. For convenience, we denote the exact and approximate MSFE as ExMSFE and AppMSFE respectively.

Derivation. Let $\mathbf{y}_t$ be an m×1 vector of variables generated by a first-order autoregressive process:

$$(1) \qquad \mathbf{y}_t = \mathbf{A}\mathbf{y}_{t-1} + \mathbf{u}_t \qquad \mathbf{u}_t \sim \mathrm{IN}(0,\Omega) \qquad t=2,\ldots,n+s$$

with $\mathbf{y}_1$ given, and where the first n observations are available for estimation and the s–period-ahead forecast is of interest. Bold characters denote vectors (if lower case) and matrices (if upper case). Although (1) appears limited to first-order processes, it is not. If the underlying process is of a higher order, it always can be "stacked" to give a first-order process. Because of that stacking, or for other reasons, the variables of interest for forecasting may be a subset (or some linear combination) of $\mathbf{y}_{n+s}$, so we introduce a selection matrix S such that $S'\mathbf{y}_{n+s}$ is the vector of interest. Further, the matrix $\mathbf{A}$ may be restricted (e.g., have zeros), so it is useful to recognize explicitly how the matrix $\mathbf{A}$ is a function of its unconstrained elements $\boldsymbol{\theta}$:

$$(2) \qquad \boldsymbol{\alpha} \equiv \mathbf{A}^{\nu} = \mathbf{R}\boldsymbol{\theta} + \mathbf{r} ,$$

where $(\cdot)^{\nu}$ denotes the column vectorizing operator, and all the elements of $\mathbf{R}$ and $\mathbf{r}$ are known.

Next, assume that $\boldsymbol{\theta}$ is estimated by $\hat{\boldsymbol{\theta}}$ which is asymptotically distributed as:

(3) $$\sqrt{n}\cdot(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \overset{D}{\rightarrow} N(0,\Psi)$$

and so

(4) $$\sqrt{n}\cdot(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) \overset{D}{\rightarrow} N(0,\Gamma)$$

where $\Gamma = \mathbf{R}\Psi\mathbf{R}'$. In finite samples, the approximate distribution of $\hat{\boldsymbol{\theta}}$ is:

(5) $$\hat{\boldsymbol{\theta}} \sim N(\boldsymbol{\theta}, \Psi/n) \ .$$

For the remainder of the derivation, (5) is treated as if it were the exact distribution of $\hat{\boldsymbol{\theta}}$, i.e., terms smaller than $O_p(n^{-1/2})$ in the distribution of $\hat{\boldsymbol{\theta}}$ are ignored.

Using the data $[\mathbf{y}_1 \ldots \mathbf{y}_n]$ to forecast $\mathbf{y}_{n+s}$ gives:

(6) $$\hat{\mathbf{y}}_{n+s} = \hat{\mathbf{A}}^{s}\mathbf{y}_n \ ,$$

the *ex ante* s–step-ahead forecast. By repeated substitution of (1) into itself at successive lags, the actual outcome $\mathbf{y}_{n+s}$ is:

(7) $$\mathbf{y}_{n+s} = \mathbf{A}^{s}\mathbf{y}_n + \sum_{i=0}^{s-1} \mathbf{A}^{i}\mathbf{u}_{n+s-i} \ ,$$

where $\mathbf{A}^0 \equiv \mathbf{I}$ if $\mathbf{A}=0$. Thus, the discrepancy between actual and forecast $\mathbf{y}_{n+s}$ is:

(8) $$(\mathbf{y}_{n+s} - \hat{\mathbf{y}}_{n+s}) = \left[\sum_{i=0}^{s-1} \mathbf{A}^{i}\mathbf{u}_{n+s-i}\right] + (\mathbf{A}^{s} - \hat{\mathbf{A}}^{s})\mathbf{y}_n \ .$$

Selecting the variable of interest gives the corresponding forecast error $\hat{g}_{n+s}$:

(9) $$\begin{aligned} \hat{g}_{n+s} &\equiv \mathbf{S}'(\mathbf{y}_{n+s} - \hat{\mathbf{y}}_{n+s}) \\ &= \mathbf{S}'\left[\sum_{i=0}^{s-1} \mathbf{A}^{i}\mathbf{u}_{n+s-i}\right] + \mathbf{S}'(\mathbf{A}^{s} - \hat{\mathbf{A}}^{s})\mathbf{y}_n \ . \end{aligned}$$

The two terms on the RHS of (9) correspond directly to the two sources of uncertainty being investigated. The first, $\mathbf{S}'\left[\sum_{i=0}^{s-1} \mathbf{A}^{i}\mathbf{u}_{n+s-i}\right]$, is the cumulation of the shocks to which $\mathbf{y}_t$ is subject over the interval $[n+1,n+s]$, where each shock is weighted by the degree to which it influences $\mathbf{S}'\mathbf{y}_{n+s}$, the variable being predicted. The second, $\mathbf{S}'(\mathbf{A}^{s} - \hat{\mathbf{A}}^{s})\mathbf{y}_n$, reflects the uncertainty present from using an estimated value of $\mathbf{A}$ rather than its true value in forecasting $\mathbf{S}'\mathbf{y}_{n+s}$. We denote these two terms $\mathbf{a}_{n,s}$ and $\mathbf{b}_{n,s}$.

Straightforwardly, the variance of the first term is:

(10) $$\mathrm{Var}(\mathbf{a}_{n,s}|\mathbf{y}_n) = \mathbf{S}'\left[\sum_{i=0}^{s-1} \mathbf{A}^i\Omega(\mathbf{A}^{i\prime})\right]\mathbf{S} \equiv \mathrm{AsyMSFE} ,$$

which is the "asymptotic" (i.e., large n) MSFE. The approximate variance of the second term is:

(11) $$\mathrm{Var}(\mathbf{b}_{n,s}|\mathbf{y}_n) = n^{-1}\cdot(\mathbf{I}\otimes\mathbf{y}_n')[\mathrm{D(s)}'\Gamma\mathrm{D(s)}](\mathbf{I}\otimes\mathbf{y}_n)$$

where $\mathrm{D(s)}' = \partial(\mathbf{S}'\mathbf{A}^s)^\nu/\partial\boldsymbol{\alpha}' = (\mathbf{S}'\otimes\mathbf{I})\left[\sum_{i=0}^{s-1} \mathbf{A}^i\otimes(\mathbf{A}^{s-i-1})'\right]$. Its derivation is more complicated and is given in Appendix A.[3] Because $\hat{\mathbf{A}}^s$ and the $\mathbf{u}_{n+s-i}$ are independent (by assumption), $\mathbf{a}_{n,s}$ and $\mathbf{b}_{n,s}$ are as well, so we can add their variances together to obtain the approximate MSFE (AppMSFE):

(12) $$\mathrm{AppMSFE}(\hat{\mathbf{g}}_{n+s}|\mathbf{y}_n) = \mathbf{S}'\left[\sum_{i=0}^{s-1} \mathbf{A}^i\Omega(\mathbf{A}^{i\prime})\right]\mathbf{S} + n^{-1}\cdot(\mathbf{I}\otimes\mathbf{y}_n')[\mathrm{D(s)}'\Gamma\mathrm{D(s)}](\mathbf{I}\otimes\mathbf{y}_n) .$$

$\mathrm{Var}(\mathbf{a}_{n,s}|\mathbf{y}_n)$ increases monotonically as s increases, but $\mathrm{Var}(\mathbf{b}_{n,s}|\mathbf{y}_n)$ may increase before decreasing to zero. Hence, $\mathrm{AppMSFE}(\hat{\mathbf{g}}_{n+s}|\mathbf{y}_n)$ may decrease as well as increase, as s increases. Equation (12) is relatively easy to implement in a computer program because it involves only sums of products of matrices. $\mathbf{S}$ and $\mathbf{y}_n$ are known, and the unknown elements of $\mathbf{A}$, $\boldsymbol{\Omega}$, and $\boldsymbol{\Gamma}$ may be replaced by consistent estimates of them.

The *univariate* AR(1) provides insight into the approximation, i.e., (1) is:

(1′) $$y_t = \beta y_{t-1} + u_t \qquad u_t \sim \mathrm{IN}(0,\sigma^2) \quad t=2,\ldots,n+s$$

with some initial condition for y_1, such as

(13) $$y_1 \sim \mathrm{IN}(0,\delta^2\sigma^2)$$

for arbitrary δ. The OLS estimator of β is asymptotically distributed as:

(4′) $$\sqrt{n}\cdot(\hat{\beta} - \beta) \xrightarrow{D} \mathrm{N}(0, [1-\beta^2]) .$$

Thus,

(14) $$\begin{aligned} \mathbf{A} &= \boldsymbol{\alpha} = \boldsymbol{\theta} = \beta \\ \boldsymbol{\Omega} &= \sigma^2 \\ \mathbf{R} &= 1 \end{aligned}$$

[3]Higher-order approximations could be obtained by employing a higher-order Tayler-series expansion in (A.5) and using distributional results in Shenton and Johnson (1965).

$$\mathbf{r} = 0$$

$$\mathbf{\Psi} = \Gamma = (1-\beta^2)$$

$$\mathbf{S} = 1$$

$$\mathbf{D}(s) = s\beta^{s-1} .$$

In that case, the forecast error (9) simplifies to:

$$(9') \qquad \hat{g}_{n+s} = (y_{n+s} - \hat{y}_{n+s}) = \sum_{i=0}^{s-1} (\beta^i)\cdot u_{n+s-i} + (\beta^s - \hat{\beta}^s)\cdot y_n ,$$

and the approximate MSFE in (12) is:

$$(12') \qquad \text{AppMSFE}(\hat{g}_{n+s}|y_n) = \sigma^2\left[\frac{1-(\beta^2)^s}{1-\beta^2}\right] + (n^{-1}\cdot y_n^2)\cdot(s\beta^{s-1})^2\cdot(1-\beta^2) .$$

Note that (12′) immediately identifies the separate contributions of the different sources of uncertainty: the first term on the RHS is the asymptotic term, the second is the part arising from coefficient uncertainty.[4]

Analytical Properties. The AppMSFE in (12′) is a function of (σ^2, y_n, β, s, n). As forecasters, we are particularly interested in knowing how and why (12′) varies as these determinants vary and in knowing how well (12′) approximates the ExMSFE. Thus, the remainder of this section considers analytical properties of the components of (12′), and the following section compares the AsyMSFE, AppMSFE, and ExMSFE numerically.

The properties of the first term on the RHS of (12′) (i.e., AsyMSFE) are relatively simple and well-known. Starting at the conditional variance of y_t (σ^2) for s=1, the AsyMSFE increases monotonically in s, tending to the *unconditional* variance of y_t ($\sigma^2/(1-\beta^2)$).[5] Because the two sources of uncertainty are additive and independent, both the AppMSFE and the ExMSFE are always larger than the AsyMSFE (with possible equality for the AppMSFE).

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[4]Although the original derivation of (12′) is difficult to ascertain, it appears as early as 1970 in Box and Jenkins (1970, p. 269).

[5]For the *one-step-ahead* forecast, (12′) simplifies to the more familiar formula (cf. Chow (1960)): $\text{AppMSFE}(\hat{g}_{n+1}|y_n) = \sigma^2 + y_n^2(1-\beta^2)/n \approx \sigma^2[1 + y_n^2(\Sigma_{t=2}^{n} y_{t-1}^2)^{-1}]$.

The structure and properties of the second term on the RHS of (12′) require some examination. Its functional form can be easily explained and interpreted via its derivation. The term $\hat{\beta}^s$ in (9′), viewed as a function of $\hat{\beta}$, is approximated by a first-order Taylor-series expansion about β to give $f(\hat{\beta}) \equiv \hat{\beta}^s = \beta^s + D(s)(\hat{\beta}-\beta) + O_p(n^{-1}) = \beta^s + s\beta^{s-1}(\hat{\beta}-\beta) + O_p(n^{-1})$. Substitution into $(\beta^s-\hat{\beta}^s)\cdot y_n$ gives $s\beta^{s-1}(\beta-\hat{\beta})\cdot y_n + O_p(n^{-1})$, from which the second RHS term in (12′) follows immediately, using (4′).[6] That term is always non-negative (and generally positive) for finite n and s, and vanishes as either s or n becomes large. However, for a given sample size n, final observed value y_n, and $\beta \neq 0$, it can either decrease monotonically as the forecast horizon s increases, or increase first and then decrease. Its path depends upon the behavior of the sequence $\{(s\beta^{s-1});\ s=0,1,2,\ldots\}$, and so upon the particular value of β. The contribution of coefficient uncertainty to the AppMSFE can be large or small relative to the latter, so the functional relationship between the AppMSFE and the forecast horizon s itself depends upon β and y_n.

To examine the behavior of the MSFE as a function of s and β, we have evaluated (12′) numerically for a range of values of (y_n, β, s, n).[7] Figures 1–6 plot the AsyMSFE, AppMSFE, and ExMSFE (stationary case) for $\beta=(0.2, 0.7, 0.9)$ in combination with $n=(10, 20)$. Both here and in following sections, the term y_n^2 in (12′) is chosen to be a "representative" value, $\sigma^2/(1-\beta^2)$, i.e., equal to its unconditional expectation.[8] The values of β imply y_t ranging from being nearly white-noise to highly autoregressive.

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[6]The effect of s on the distribution of $\hat{\beta}^s$ also can be seen through the following analogy with a standardized normal variate x. $Var(x^s) = (2s)!/(2^s \cdot s!)$, e.g., $Var(x^s) = 1, 3, 15, 105$ for $s = 1, 2, 3, 4$. Clearly, taking a power of $\hat{\beta}$ can increase its variance dramatically.

[7]Because σ^2 is a scale factor in (1′) (and hence (12′)), we can set $\sigma^2=1$ without loss of generality: in that case, y_n is measured in standard deviations of u_t.

[8]Another justification for this choice is that the unconditional expectation $E[(\beta^s-\hat{\beta}^s)^2 y_n^2]$ is approximately $E[(\beta^s-\hat{\beta}^s)^2]\cdot E[y_n^2]$ because $\hat{\beta}$ and y_n are approximately independent. See Phillips (1979) for an extensive discussion on the conditional and unconditional finite sample distributions of the forecast error.

The figures reveal three distinct patterns which depend upon the values of β and n: steadily decreasing AppMSFE, steadily increasing AppMSFE, and an AppMSFE which increases and then decreases. For small β, the AppMSFE is declining almost uniformly as the horizon increases, with the initial (one-step-ahead) AppMSFE being the *largest*. That arises because the uncertainty from estimating β is large (from (4′)), but that uncertainty is unimportant in forecasting y_{n+s} except for s=1: mathematically, $s\beta^{s-1}$ in (12′) is approximately zero except for s=1, when it is approximately unity. Because the AsyMSFE changes little as s increases, the AppMSFE is declining from the start (Figures 1 and 2).

For larger values of β, the variance from coefficient uncertainty increases first and then falls because the multiplicative coefficient s in $s\beta^{s-1}$ dominates for small s but the exponential term (β^{s-1}) dominates for large s. As the sum of two components, one monotonically increasing and the other increasing and then tending to zero, the AppMSFE can either increase first and then fall towards the unconditional variance of y_t (Figures 3–6) or increase monotonically, approaching that asymptotic variance (also increasing in s) from *above*. The latter would be the case for *all* the figures if the sample size n were large enough. Thus, the potentially large and varying contribution of coefficient uncertainty to the MSFE explains the puzzling phenomena that HMP (pp. 333–335) note on the behavior of the exact MSFE as s increases; cf. Chong and Hendry (1986, p. 685). The second component of the AppMSFE provides a simple analytical explanation of this behavior, to the extent that the AppMSFE offers a good approximation to the ExMSFE. Although there are some notable discrepancies between the AppMSFE and the ExMSFE in the figures (primarily for n=10 with s=4), the AppMSFE does remarkably well in approximating the ExMSFE, so well that it is difficult to distinguish them at n=20. The accuracy of approximation is the focus of Section 3, which compares the AsyMSFE, AppMSFE, and ExMSFE numerically for a range of values of (σ^2, y_n, β, s, n). Before doing so, we note some empirical implications of these results.

Estimated autoregressive coefficients in dynamic econometric models range from the very small (e.g., for equations in first differences) to those close to unity (e.g., for equations

MEAN-SQUARE FORECAST ERROR

Figure 1. β=.2 n=10

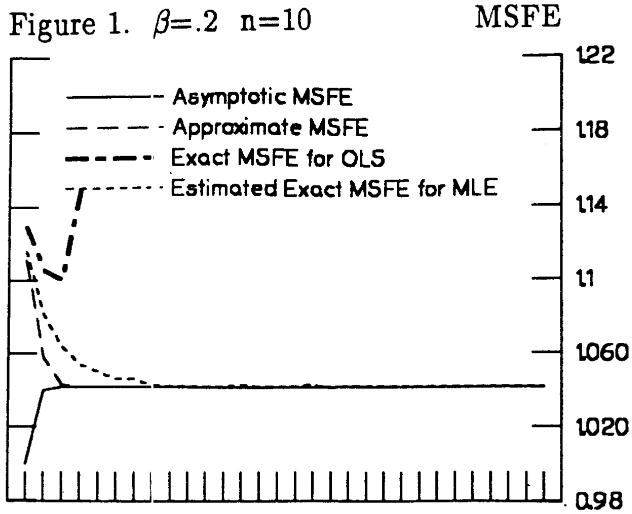


Figure 2. β=.2 n=20

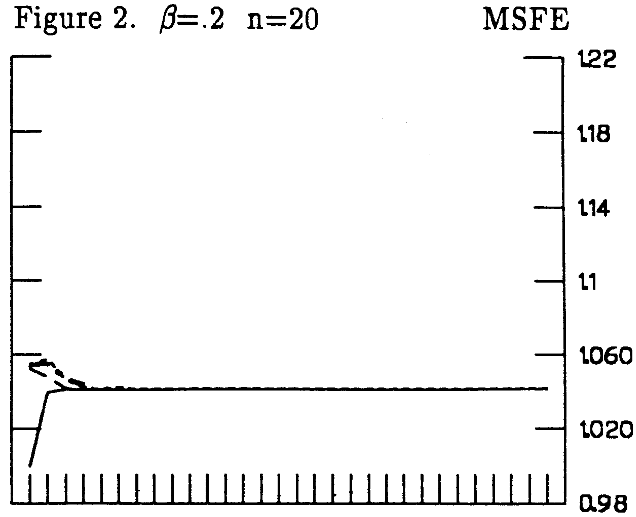


Figure 3. β=.7 n=10

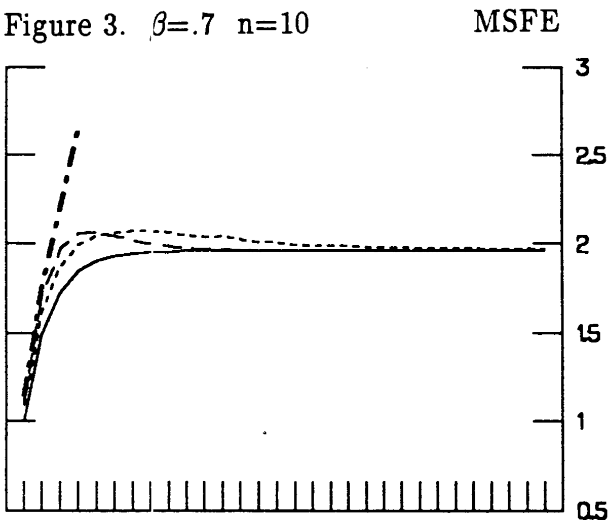


Figure 4. β=.7 n=20

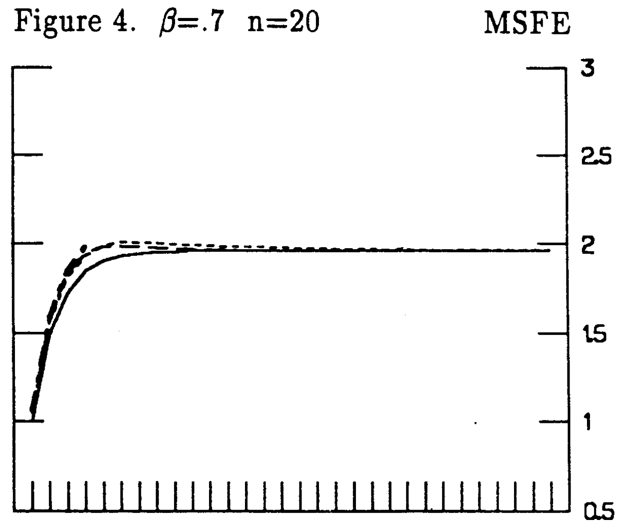


Figure 5. β=.9 n=10

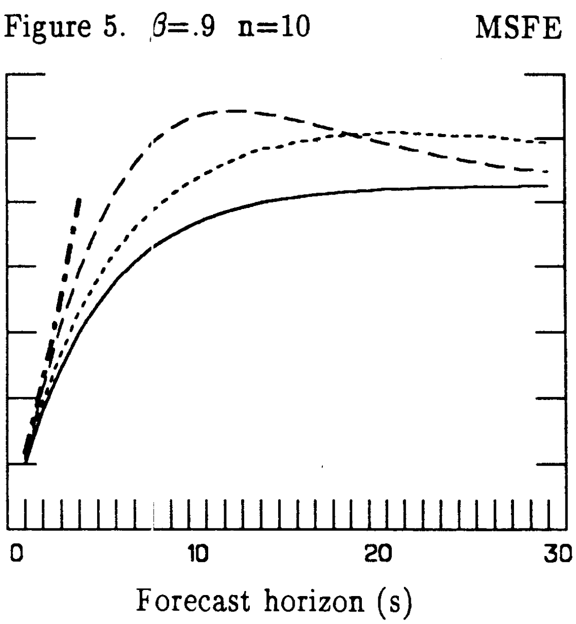
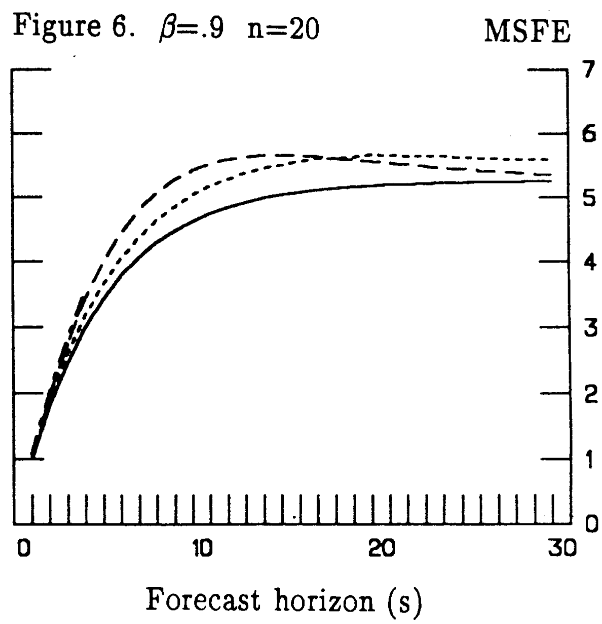


Figure 6. β=.9 n=20



in levels). Even for the corresponding (and wide) range of values for β, the maximum of the AppMSFE is often between one and twelve periods, precisely the range over which we are interested in forecasting most accurately. That is also the range for which the AsyMSFE appears the poorest approximation to the approximate and exact MSFE. In fact, the AsyMSFE generally underestimates the AppMSFE (and ExMSFE) for finite horizons, and the former need not even be the main component of the latter.

3. Numerical Properties of the Asymptotic, Approximate, and Exact MSFE

This section contrasts the AsyMSFE and AppMSFE with the ExMSFE derived by HMP and with Monte Carlo estimates of the ExMSFE calculated by Orcutt and Winokur (1969). In both of these studies, the AsyMSFE captures much of the variation across experiments, and the AppMSFE does even better in doing so. The inaccuracy of approximation appears related to how close conditions for the existence of the ExMSFE are to being violated, so we discuss the existence of the ExMSFE in the context of both papers.

HMP. Taking advantage of the explicit relationship between the OLS estimator and the disturbances $\{u_t\}$, HMP derive an analytical expression for the exact MSFE for the univariate AR(1) process given by (1′)+(13) for OLS and tabulate it for two cases: (a) $\delta^2=(1-\beta^2)^{-1}$ (the "stationary case") and (b) $\delta^2=1$ (the "non-stationary case"). Tables 1a and 2a give the percent discrepancies between the AsyMSFE in (10) and the associated ExMSFE from HMP (Tables 1 and 2) under each of those assumptions; Tables 1b and 2b likewise give percent discrepancies between the *approximate* and exact MSFE.[9] Because (10) holds exactly, discrepancies between the approximate and exact MSFE arise because (a) the asymptotic and finite sample variances of $\hat{\beta}$ differ (and hence so do the respective variances of $\hat{\beta}^s$ in (9′)), (b) $\hat{\beta}$ is biased in finite samples but not asymptotically (likewise for $\hat{\beta}^s$), (c) the first-order Taylor-series approximation of $\hat{\beta}^s$ about β^s ignores important terms, and (d) the approximate MSFE is conditional upon y_n whereas the exact MSFE treats y_n

[9]Because only (n–1) observations are actually used in estimating β, the calculation of (12′) for the tables uses (n–1) rather than n.

Appendix B describes the calculations here and with respect to Orcutt and Winokur's experiments, and lists the corresponding values of AsyMSFE and AppMSFE.

Percent Deviations of the Asymptotic and Approximate MSFE from the Exact MSFE: Stationary Case

Table 1a: 100·[1 − (AsyMSFE/ExMSFE)].

		β											
n	s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
10	1	11.4	11.4	11.4	11.5	11.7	11.9	12.2	12.5	13.1	13.6	13.4	11.9
10	2	4.7	5.0	5.9	7.3	9.1	11.1	13.3	15.8	18.5	21.3	22.1	20.9
10	3	3.7	4.1	5.3	7.4	10.2	13.6	17.4	21.5	25.9	30.0	31.2	29.5
10	4	7.6	8.0	9.3	11.6	15.1	19.5	24.8	30.5	36.0	40.6	41.5	38.4
15	1	7.1	7.1	7.1	7.1	7.2	7.3	7.5	7.7	8.0	8.6	8.7	8.0
15	2	1.6	1.8	2.4	3.3	4.5	5.9	7.4	9.0	10.9	13.3	14.5	14.2
15	3	.6	.7	1.2	2.1	3.5	5.3	7.6	10.3	13.6	17.7	19.8	19.9
15	4	.3	.4	.7	1.4	2.7	4.7	7.7	11.6	16.4	22.2	25.1	25.4
20	1	5.2	5.2	5.2	5.2	5.2	5.3	5.4	5.5	5.8	6.2	6.5	6.1
20	2	.8	1.0	1.4	2.2	3.1	4.1	5.2	6.4	7.9	9.7	10.8	10.9
20	3	.2	.3	.6	1.1	2.0	3.2	4.8	6.8	9.2	12.5	14.6	15.3
20	4	.1	.1	.3	.6	1.3	2.5	4.3	6.9	10.4	15.2	18.2	19.4
25	1	4.1	4.1	4.1	4.1	4.1	4.2	4.2	4.3	4.5	4.8	5.1	4.9
25	2	.5	.6	1.0	1.6	2.3	3.2	4.1	5.1	6.2	7.6	8.7	8.9
25	3	.1	.2	.4	.7	1.4	2.3	3.6	5.1	7.1	9.7	11.6	12.5
25	4	.0	.0	.1	.3	.8	1.6	3.0	4.9	7.7	11.6	14.3	15.8

Table 1b: 100·[1 − (AppMSFE/ExMSFE)].

		β											
n	s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
10	1	1.5	1.5	1.6	1.7	1.9	2.1	2.4	2.8	3.4	4.0	3.7	2.1
10	2	4.7	4.6	4.3	3.9	3.5	3.2	3.1	3.5	4.4	5.6	5.6	3.4
10	3	3.7	4.1	5.2	6.7	8.3	9.5	10.2	10.6	11.1	11.4	10.6	6.4
10	4	7.6	8.0	9.3	11.5	14.5	17.9	20.8	22.6	23.1	21.9	19.4	11.8
15	1	.4	.4	.5	.5	.6	.7	.8	1.1	1.5	2.0	2.2	1.4
15	2	1.6	1.5	1.3	1.1	.7	.5	.4	.5	1.0	2.2	2.9	2.1
15	3	.6	.7	1.1	1.7	2.2	2.4	2.4	2.3	2.5	3.6	4.3	3.1
15	4	.3	.4	.7	1.3	2.3	3.4	4.5	5.1	5.5	6.4	6.9	4.7
20	1	.2	.2	.2	.2	.3	.3	.4	.6	.8	1.3	1.5	1.1
20	2	.8	.8	.7	.5	.2	.1	−.1	.0	.3	1.2	1.9	1.6
20	3	.2	.3	.5	.8	1.0	1.0	.9	.7	.7	1.5	2.5	2.2
20	4	.1	.1	.2	.5	1.0	1.5	1.9	1.9	1.9	2.5	3.5	3.0
25	1	.1	.1	.1	.1	.1	.2	.2	.3	.5	.9	1.2	.9
25	2	.5	.5	.4	.2	.1	−.1	−.1	−.1	.1	.7	1.4	1.4
25	3	.1	.2	.3	.5	.6	.6	.4	.2	.1	.7	1.6	1.8
25	4	.0	.0	.1	.3	.6	.8	1.0	.9	.7	1.1	2.1	2.2

Percent Deviations of the Asymptotic and Approximate MSFE from the Exact MSFE: Non–Stationary Case

Table 2a: 100·[1 − (AsyMSFE/ExMSFE)].

		β											
n	s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
10	1	11.4	11.4	11.5	11.7	11.9	12.3	12.8	13.6	14.9	16.9	18.1	19.3
10	2	4.7	5.1	6.0	7.4	9.3	11.6	14.2	17.3	21.2	26.1	29.0	31.5
10	3	3.7	4.1	5.4	7.6	10.6	14.3	18.8	24.0	30.1	37.2	41.1	44.4
10	4	7.6	8.1	9.5	12.0	15.8	21.0	27.3	34.5	42.2	50.4	54.5	57.8
15	1	7.1	7.1	7.1	7.2	7.3	7.5	7.7	8.1	8.9	10.3	11.5	12.6
15	2	1.6	1.8	2.4	3.4	4.6	6.0	7.7	9.6	12.1	15.9	18.5	21.1
15	3	.6	.7	1.3	2.2	3.6	5.5	7.9	11.1	15.2	21.2	25.2	29.0
15	4	.3	.4	.7	1.4	2.8	4.9	8.1	12.6	18.6	26.9	32.2	37.0
20	1	5.2	5.2	5.2	5.2	5.3	5.4	5.5	5.8	6.3	7.3	8.3	9.4
20	2	.8	1.0	1.4	2.2	3.1	4.2	5.4	6.7	8.5	11.3	13.6	16.0
20	3	.2	.3	.6	1.1	2.0	3.3	5.0	7.2	10.1	14.6	18.2	21.9
20	4	.1	.1	.3	.6	1.3	2.5	4.5	7.4	11.5	17.9	22.7	27.6
25	1	4.1	4.1	4.1	4.1	4.2	4.2	4.3	4.5	4.8	5.6	6.4	7.5
25	2	.5	.6	1.0	1.6	2.4	3.2	4.2	5.2	6.6	8.7	10.6	13.0
25	3	.1	.2	.4	.8	1.4	2.3	3.6	5.3	7.6	11.1	14.2	17.8
25	4	.0	.0	.1	.3	.8	1.6	3.1	5.2	8.3	13.3	17.5	22.3

Table 2b: 100·[1 − (AppMSFE/ExMSFE)].

		β											
n	s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
10	1	1.5	1.6	1.7	1.8	2.1	2.5	3.1	4.0	5.5	7.6	9.0	10.4
10	2	4.7	4.6	4.4	4.0	3.7	3.7	4.1	5.2	7.5	11.4	14.0	16.4
10	3	3.7	4.1	5.2	6.9	8.6	10.3	11.7	13.5	16.1	20.5	23.4	26.2
10	4	7.6	8.1	9.5	11.9	15.3	19.3	23.4	27.0	30.6	34.8	37.3	39.6
15	1	.4	.4	.5	.5	.7	.8	1.1	1.6	2.4	3.9	5.1	6.4
15	2	1.6	1.5	1.3	1.1	.8	.6	.7	1.1	2.3	5.1	7.4	9.9
15	3	.6	.7	1.2	1.7	2.2	2.6	2.8	3.1	4.3	7.7	10.8	14.1
15	4	.3	.4	.7	1.4	2.4	3.6	5.0	6.2	8.0	12.1	15.7	19.5
20	1	.2	.2	.2	.2	.3	.4	.6	.8	1.3	2.4	3.4	4.6
20	2	.8	.8	.7	.5	.3	.1	.1	.3	1.0	2.9	4.9	7.3
20	3	.2	.3	.5	.8	1.0	1.1	1.1	1.1	1.6	3.9	6.6	9.8
20	4	.1	.1	.2	.5	1.0	1.6	2.0	2.4	3.0	5.6	8.9	12.9
25	1	.1	.1	.1	.1	.2	.2	.3	.5	.8	1.6	2.5	3.6
25	2	.5	.5	.4	.3	.1	.0	−.1	.0	.5	1.9	3.6	5.8
25	3	.1	.2	.3	.5	.6	.6	.5	.4	.6	2.3	4.5	7.7
25	4	.0	.0	.1	.3	.6	.9	1.1	1.2	1.3	3.0	5.8	9.7

as stochastic.[10] Discrepancies between the asymptotic and exact MSFE arise because $\hat{\beta}$ is not identically β.

The asymptotic formula captures the behavior of the ExMSFE well for small to medium values of β and for large n, with deviations of the order of 5–15%, but it does poorly otherwise. The AppMSFE fares better: typical departures are 2–3% or less. As with the AsyMSFE, more sizable discrepancies appear for large s paired with small n and/or large β: the concept of "effective sample size" predicts the possibility of such departures under those conditions (cf. Sims (1974) and Hendry (1984)). Almost invariably, the AppMSFE is smaller than the exact MSFE.

The accuracy and the generality of the approximate MSFE do not in any way belittle HMP's exact results for the AR(1) process. To the contrary, exact results are highly desirable *because* they involve no approximation error; and they are essential for assessing the accuracy of approximations such as (12′).

Orcutt and Winokur. In a Monte Carlo study evaluating numerous facets to least-squares estimation of the univariate AR(1) process, Orcutt and Winokur (1969) estimate the ExMSFE for all combinations of β=(0.0, 0.3, 0.6, 0.9, 1.0), n=(10, 20, 40), and s=(1, 2, 3, 4). Unlike HMP, they include a constant term in the estimation of the AR(1) process: that is easily incorporated into the Schmidt-Baillie approximation by including a non-stochastic variable in the vector $\mathbf{y}_t$ which is equal to its own lag and is initialized at unity. Because of the additional uncertainty introduced by estimating a constant, the resulting AppMSFE is always larger than that for equations with a known constant, even if (as in Orcutt and Winokur's experiments) the constant is zero. Equation (3) is *not* valid for Orcutt and Winokur's experiment with β=1, so we use β=0.9999 instead. That should (and does) offer a good approximation, given the difficulty in finite samples in distinguishing between a unit root and a root close to (but less than) unity.

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[10]In actual forecasting, y_n is given, in which case the conditional MSFE seems more appropriate than the unconditional MSFE. At another level, y_n is often subject to data revisions, so it may be invalid to treat its (latent) value as known. This exemplifies another source of uncertainty and it is outside our analysis.

Tables 3a and 3b respectively list the percent discrepancies of the AsyMSFE and AppMSFE from the estimated exact MSFE (Ex$\hat{\text{M}}$SFE) of Orcutt and Winokur (Table VII). The overall pattern parallels that in Tables 1 and 2: the AppMSFE generally fares better than the AsyMSFE, with the latter almost invariably underestimating the estimated exact MSFE. Both the asymptotic and approximate MSFE fare better for larger n and smaller β. Orcutt and Winokur's estimates of the exact MSFE are subject to sampling errors from the Monte Carlo simulation. As Appendix C shows, the standard error for their estimated ExMSFE is about 4.5%, so the discrepancies between their estimates and the AppMSFE appear to be due almost exclusively to simulation uncertainty.[11]

Existence of the MSFE. HMP show that the ExMSFE exists for the AR(1) model if and only if the forecast horizon s is not greater than (n–2)/2 [not greater than (n–3)/2 if a constant term is included in the regression, cf. Magnus and Pesaran (1989)]. For instance, for n=10 in Tables 1 and 2, the ExMSFE exists for s≤4 only. The worsening of the approximation error as s increases may be due to the declining number of moments of the forecast error. The effects of the existence of moments are also suggested by Tables 1–2 and Figures 1–6, where the convergence of the exact and approximate MSFE appears faster than $O(n^{-2})$ for large s.

Interestingly, the ExMSFE does not exist for Orcutt and Winokur's experiments with (n=10, s=4), yet the AppMSFE still does quite well at approximating their estimates. We interpret this surprising result as follows. For values of s for which the ExMSFE does not exist, the AppMSFE still can be calculated and may provide accurate confidence intervals for the forecasts. However, because there is a significant probability of $\hat{\beta}$ being greater than unity and thus causing the forecast error to explode for large s, the *tails* of the exact density of the forecast error are too thick for its variance to exist. Sargan (1982) examines a similar situation in which an estimator is well-behaved asymptotically but has

[11]The accuracy of AppMSFE for the *univariate* AR(1) processes in HMP and Orcutt and Winokur (1969) adds to Chong and Hendry's (1986) Monte Carlo evidence on the accuracy of the Baillie-Schmidt approximation for a two-equation model.

Percent Deviations of the Asymptotic and Approximate MSFE
from Orcutt and Winokur's (1969) Monte Carlo Estimates of the Exact MSFE

Table 3a: $100 \cdot [1 - (\text{AsyMSFE}/\text{Ex}\hat{\text{M}}\text{SFE})]$.

		β				
n	s	0.00	0.30	0.60	0.90	1.00
10	1	25.4	23.5	23.8	22.9	22.7
10	2	9.6	25.7	26.8	29.6	27.4
10	3	20.7	24.2	34.0	35.2	35.4
10	4	18.4	24.0	42.0	41.5	43.4
20	1	4.7	15.8	9.6	16.0	13.0
20	2	10.3	8.6	11.1	23.8	22.3
20	3	3.3	2.3	14.2	23.5	23.3
20	4	.1	9.0	18.8	24.0	24.7
40	1	11.3	.0	5.6	8.2	9.5
40	2	7.0	2.1	5.4	10.3	8.0
40	3	9.2	10.0	4.0	11.1	10.6
40	4	−2.2	4.8	8.6	14.9	16.2

Table 3b: $100 \cdot [1 - (\text{AppMSFE}/\text{Ex}\hat{\text{M}}\text{SFE})]$.

		β				
n	s	0.00	0.30	0.60	0.90	1.00
10	1	8.9	6.5	6.9	5.8	5.5
10	2	−.5	10.2	2.9	.0	−4.9
10	3	11.9	8.8	9.3	−3.5	−7.7
10	4	9.4	8.4	19.0	−2.6	−6.9
20	1	−5.4	6.9	.1	7.1	3.9
20	2	5.6	−.4	−2.6	8.6	5.9
20	3	−1.8	−7.1	−.9	1.9	−1.0
20	4	−5.2	.2	3.6	−3.1	−7.0
40	1	6.7	−5.1	.7	3.5	4.9
40	2	4.6	−2.7	−1.8	1.6	−1.4
40	3	6.8	5.8	−4.3	−1.2	−3.2
40	4	−4.9	.3	.2	.1	−1.0

no moments in finite samples. In light of his paper, the approximation in (12) may be interpreted as analogous to the Nagar approximation for the moments of an estimator. Conversely, the lack of existence of the ExMSFE (when that occurs) *must* be due to terms smaller than $O_p(n^{-1})$ in the squared forecast error (i.e., $o_p(n^{-1})$ and probably $O_p(n^{-1.5})$) because the AppMSFE accounts for all terms $O_p(n^{-1})$ and larger, and it exists for all s. The existence or otherwise of the ExMSFE at relatively large s is an issue recurring in the following section.

4. The MSFE for the Maximum Likelihood Estimator

Because the Schmidt-Baillie approximation relies on only the *asymptotic* distribution of the estimator used, the formulae in Section 2 for OLS are equally valid for all estimators asymptotically equivalent to OLS (cf. (4) and (4′) above). Exact maximum likelihood (ML) is such an estimator, and one which has several desirable features in the present context.[12] In particular, its ExMSFE exists at all forecast horizons s, independent of the size of the estimation period n. We show this by examining the properties of the ExMSFE for truncated estimators: truncation at an arbitrary value implies the existence of the ExMSFE at all finite forecast horizons, and truncation at the unit circle (as with ML) implies a bound on the ExMSFE, independent of the forecast horizon. The first of these results allows us to assess the generality of the Schmidt-Baillie approximation at forecast horizons longer than those feasible for OLS. However, because the analytical formula for the ExMSFE with ML is unknown, we compare the AppMSFE with Monte Carlo estimates of the exact MSFE. Again, the AppMSFE is a remarkably good approximation, even for short estimation periods and long forecast horizons.

Truncated estimators of β and existence of the exact MSFE. Consider a truncated estimator $\bar{\beta}$ such that $|\bar{\beta}| \leq \gamma$ for some positive bound γ. The corresponding forecast error is:

$$(9'') \qquad \bar{g}_{n+s} = \sum_{i=0}^{s-1} (\beta^i) \cdot u_{n+s-i} + (\beta^s - \bar{\beta}^s) \cdot y_n .$$

[12]Cf. Maekawa (1987) who shows an equivalence to $O(n^{-1})$ between the *distributions* of the forecast error for OLS and approximate ML.

A bound can be placed on the MSFE for $\bar{g}_{n+s}$ by application of the triangle and Schwartz inequalities and by noting that $E(y_n^2)=\sigma^2/(1-\beta^2)$.

$$(15) \qquad \begin{aligned} \mathrm{ExMSFE}(\bar{g}_{n+s}|y_n) &= \sigma^2\left[\frac{1-(\beta^2)^s}{1-\beta^2}\right] + E[(\beta^s-\bar{\beta}^s)^2\cdot y_n^2] \\ &\leq \sigma^2\left[\frac{1-(\beta^2)^s}{1-\beta^2}\right] + E[(\beta^s-\bar{\beta}^s)^2]\cdot E(y_n^2) \\ &\leq \sigma^2\left[\frac{1-(\beta^2)^s}{1-\beta^2}\right] + E[(|\beta^s|+|\bar{\beta}^s|)^2]\cdot E(y_n^2) \\ &\leq \sigma^2\left[\frac{1-\beta^{2s}+(1+\gamma^s)^2}{1-\beta^2}\right] \end{aligned}$$

If the bound is the unit circle (γ=1), then a slightly looser bound exists which is independent of s: i.e., $5\sigma^2/(1-\beta^2)$, five times the unconditional variance of the process. The existence of the ExMSFE does not require that the estimator be consistent for any value whatsoever. Conversely, because neither bound makes use of the asymptotic properties of $\bar{\beta}$, neither converges to the AsyMSFE as $n\to\infty$. Even so, the existence of a bound (and so of the ExMSFE) indicates how sensitive the existence conditions are to minor changes in the distributional assumptions of the estimator being used.[13]

The MSFE for Maximum Likelihood. In order to assess the accuracy of the AppMSFE in approximating the ExMSFE for ML without the advantage of exact analytical formulae, we have estimated the exact MSFE by Monte Carlo for a wide range of β, n, and s, and compared those estimates with the asymptotic and approximate MSFE. Specifically, we chose β = (0.0, 0.1, 0.2, ..., 0.8, 0.9, 0.95, 0.99), n = (10, 15, 20, 25, 40), and s = (1, 2, ..., 30), with $\delta^2=\sigma^2/(1-\beta^2)$ to ensure stationarity. This design embeds the range of values evaluated by HMP for OLS with a stationary AR(1) process. However, because the ExMSFE for ML exists for all forecast horizons, we can compare its values with

[13]E.g., the truncated estimator based on OLS and with $\gamma=10^{310}$ (the range permitted by double-precision calculations on a computer) implies existence of the ExMSFE for all s, yet that truncated estimator will look like OLS for virtually all practical purposes.

the AppMSFE at much longer forecast horizons than available to HMP. To obtain reasonably accurate Monte Carlo estimates, we used 10,000 replications per experiment and in addition implemented a control variate; Appendix D provides details.

Tables 4a and 4b respectively list the percent deviations of the AsyMSFE and the AppMSFE from the control variate "pooled" estimates of the exact MSFE for ML (PoMSFE) with n=10. Tables 5a–b, 6a–b, 7a–b, and 8a–b likewise list percent deviations for n = 15, 20, 25, 40. To condense presentation, values for s>10 appear for s at multiples of five: at long horizons, the exact, asymptotic, and approximate MSFE all change slowly as a function of s in any case.[14] As with OLS, the AsyMSFE does reasonably well for small to medium values of β and for large n, with the AppMSFE doing better, and over a wider range of β and n. As n increases, both deviations generally decline, as would be expected because the ExMSFE is tending to the AsyMSFE. Unlike with OLS, the AppMSFE often *over*-estimates the ExMSFE for β in the range of [0.80, 0.95]. However, for β very close to the unit circle (β=0.99), the AppMSFE again under-estimates the ExMSFE. The boundedness of the ML estimator is affecting the ExMSFE for large β, but little more can be said without considering terms smaller than $O(n^{-1})$ in the MSFE. Even so, the AppMSFE offers a remarkably simple and accurate summary of the behavior of the *exact* MSFE for ML.

Figures 1–6 graph the estimated values of the ExMSFE for ML as well as the ExMSFE for OLS, the AsyMSFE, and the AppMSFE. For β=0.2 with n=10, the AppMSFE approximates the ExMSFE for ML better than it does the ExMSFE for OLS. At medium and large values of β (0.7 and 0.9) with n=10, the AppMSFE still does well, but it over-estimates the ExMSFE at short to medium horizons and under-estimates it at very long horizons. Relatedly, the "hump" so evident for the AppMSFE is less pronounced (but still present) for the ExMSFE for ML. For n=20, the deviations between the exact and approximate MSFE are much smaller than for n=10, as expected.

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[14] *All values* of the respective MSFEs for s≤30 are tabulated in Appendix D.

Percent Deviations of the Asymptotic and Approximate MSFE
from Pooled Monte Carlo Estimates of the Exact MSFE for Maximum Likelihood (n=10)

Table 4a: 100·[1 − (AsyMSFE/PoMSFE)].

s	β 0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	10.4	9.6	10.4	9.4	9.6	10.0	10.9	8.3	8.5	6.7	6.3	14.4
2	3.3	3.8	3.8	4.2	5.9	6.1	7.6	7.9	8.0	7.5	9.1	24.7
3	1.1	1.9	2.1	3.3	4.2	5.7	7.3	7.8	8.0	8.5	11.7	33.2
4	.7	1.2	1.2	1.7	3.1	3.7	6.2	7.1	7.7	9.5	14.1	39.7
5	.4	.2	.8	1.0	2.5	3.3	5.0	6.8	7.9	10.0	16.1	45.3
6	.1	.6	.5	.5	2.0	3.3	3.9	6.3	7.7	11.0	18.3	49.1
7	.1	.4	.4	.6	1.6	2.8	3.3	5.9	8.1	11.7	20.4	52.9
8	.1	.5	.2	.3	.7	2.2	3.2	5.6	7.6	11.9	22.6	56.3
9	.1	.1	.0	.3	.8	1.3	2.3	5.2	7.7	12.2	24.3	59.3
10	.0	.1	.1	.2	.8	1.2	1.9	4.3	7.5	12.6	25.5	61.8
15	.0	.0	.0	.1	.3	.4	.9	2.3	5.2	14.0	32.4	70.7
20	.1	.0	.0	.0	.0	.1	.3	1.0	3.2	14.4	37.0	76.4
25	.0	.0	.0	.0	.0	.1	.1	.6	2.1	13.2	39.2	80.2
30	.0	.0	.0	.0	.0	.1	.0	.5	1.7	11.3	42.1	82.7

Table 4b: 100·[1 − (AppMSFE/PoMSFE)].

s	β 0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	.4	-.4	.4	-.7	-.4	.0	1.0	-1.9	-1.7	-3.7	-4.2	4.9
2	3.3	3.3	2.2	.7	.1	-2.3	-3.2	-5.6	-8.0	-10.8	-10.1	8.2
3	1.1	1.9	2.0	2.5	2.1	1.2	-.8	-5.0	-10.4	-15.9	-14.8	11.4
4	.7	1.2	1.2	1.6	2.5	1.7	1.1	-3.4	-11.0	-19.0	-18.4	13.7
5	.4	.2	.8	1.0	2.4	2.5	2.1	-1.0	-9.4	-21.4	-21.5	16.2
6	.1	.6	.5	.5	1.9	3.0	2.4	.8	-7.7	-21.8	-23.2	16.9
7	.1	.4	.4	.6	1.5	2.7	2.5	2.3	-4.9	-21.8	-24.1	18.5
8	.1	.5	.2	.3	.7	2.2	2.9	3.3	-3.1	-21.5	-24.2	20.1
9	.1	.1	.0	.3	.8	1.3	2.1	3.8	-.8	-20.6	-24.2	21.7
10	.0	.1	.1	.2	.8	1.2	1.8	3.5	.8	-18.9	-24.4	23.2
15	.0	.0	.0	.1	.3	.4	.9	2.2	3.5	-8.4	-17.5	28.4
20	.1	.0	.0	.0	.0	.1	.3	1.0	2.9	1.0	-7.5	33.5
25	.0	.0	.0	.0	.0	.1	.1	.6	2.0	5.9	1.2	37.4
30	.0	.0	.0	.0	.0	.1	.0	.5	1.7	7.6	12.0	40.4

Percent Deviations of the Asymptotic and Approximate MSFE
from Pooled Monte Carlo Estimates of the Exact MSFE for Maximum Likelihood (n=15)

Table 5a: 100·[1 − (AsyMSFE/PoMSFE)].

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	6.1	7.3	7.7	6.6	6.7	6.3	6.7	6.7	5.8	4.9	4.0	13.2
2	1.4	1.7	1.8	3.3	3.9	3.6	6.0	5.9	6.6	6.8	6.5	23.4
3	.3	.6	.9	2.3	2.9	3.5	5.1	5.9	6.7	6.9	8.7	31.3
4	.3	.4	.4	.5	1.6	3.0	3.8	5.8	6.7	7.9	10.7	38.2
5	.0	.2	.3	.5	1.2	2.3	3.4	4.7	6.8	8.7	12.6	43.2
6	.1	.0	.0	.4	.6	1.4	3.2	4.9	6.4	8.9	14.2	48.0
7	.0	.0	.0	.1	.4	1.1	2.6	4.1	5.8	9.4	16.0	51.6
8	.0	.1	.0	.1	.2	.7	1.7	4.0	5.9	9.5	17.4	55.0
9	.0	.0	.0	.1	.2	.5	1.5	3.4	6.1	9.6	18.8	57.8
10	.0	.0	.0	.1	.2	.7	1.3	3.2	5.9	9.6	19.7	60.3
15	.0	.0	.0	.0	.1	.2	.5	1.2	3.3	9.3	24.8	69.5
20	.0	.0	.0	.0	.0	.1	.1	.4	2.2	8.1	28.1	75.0
25	.0	.0	.0	.0	.0	.1	.1	.1	1.2	8.0	31.3	78.9
30	.0	.0	.0	.0	.0	.0	.0	.1	.5	7.5	33.1	81.6

Table 5b: 100·[1 − (AppMSFE/PoMSFE)].

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	-.6	.6	1.1	-.1	.1	-.4	.1	.0	-.9	-1.9	-2.9	7.0
2	1.4	1.4	.7	1.0	.1	-1.9	-1.1	-3.0	-3.8	-5.2	-6.2	12.6
3	.3	.6	.8	1.8	1.5	.5	-.2	-2.4	-5.2	-9.1	-8.9	16.9
4	.3	.4	.4	.5	1.2	1.6	.5	-1.0	-5.4	-10.7	-11.0	21.0
5	.0	.2	.3	.5	1.1	1.8	1.5	-.4	-4.4	-11.7	-12.5	23.7
6	.1	.0	.0	.4	.6	1.2	2.2	1.3	-3.6	-12.7	-13.9	26.8
7	.0	.0	.0	.1	.3	1.0	2.1	1.7	-2.7	-12.6	-14.2	28.9
8	.0	.1	.0	.1	.2	.7	1.5	2.5	-1.1	-12.6	-14.6	31.0
9	.0	.0	.0	.1	.2	.5	1.4	2.5	.5	-12.1	-14.6	32.7
10	.0	.0	.0	.1	.2	.7	1.2	2.6	1.5	-11.3	-14.9	34.5
15	.0	.0	.0	.0	.1	.2	.5	1.2	2.2	-5.8	-10.9	41.3
20	.0	.0	.0	.0	.0	.1	.1	.4	2.0	-1.2	-4.6	45.6
25	.0	.0	.0	.0	.0	.1	.1	.1	1.1	3.0	3.7	49.6
30	.0	.0	.0	.0	.0	.0	.0	.1	.5	4.9	10.7	52.6

Percent Deviations of the Asymptotic and Approximate MSFE
from Pooled Monte Carlo Estimates of the Exact MSFE for Maximum Likelihood (n=20)

Table 6a: 100·[1 − (AsyMSFE/PoMSFE)].

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	5.4	5.1	5.2	6.1	5.6	5.5	4.0	4.4	4.4	2.8	3.7	11.0
2	.8	.7	1.7	1.7	3.0	3.7	3.7	4.6	5.7	4.6	5.9	21.1
3	.0	.3	.5	.9	1.8	2.8	3.1	4.9	5.9	5.5	7.5	29.1
4	.1	.1	.2	.4	.8	1.9	2.5	4.4	5.9	5.7	9.4	35.6
5	.0	.0	.1	.2	.5	1.4	1.9	4.1	5.4	6.2	10.8	40.4
6	.0	.0	.1	.1	.3	1.0	1.4	3.8	5.3	6.6	11.7	45.0
7	.0	.0	.0	.0	.2	.5	1.3	3.0	5.3	7.2	13.3	48.7
8	.0	.0	.0	.0	.1	.4	1.1	2.5	5.3	7.6	14.1	52.1
9	.0	.0	.0	.0	.1	.3	.8	1.9	5.4	7.7	14.9	55.1
10	.0	.0	.0	.0	.0	.2	.5	1.7	5.0	8.0	16.1	57.5
15	.0	.0	.0	.0	.0	.1	.2	.8	2.8	8.5	20.5	66.9
20	.0	.0	.0	.0	.0	.0	.0	.4	1.7	8.6	24.1	72.7
25	.0	.0	.0	.0	.0	.0	.0	.1	.9	6.8	26.4	76.9
30	.0	.0	.0	.0	.0	.0	.0	.1	.4	6.0	27.7	80.0

Table 6b: 100·[1 − (AppMSFE/PoMSFE)].

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	.4	.2	.2	1.2	.6	.6	-1.0	-.7	-.6	-2.3	-1.3	6.3
2	.8	.5	.9	.0	.2	-.3	-1.6	-2.0	-2.1	-4.4	-3.5	12.9
3	.0	.3	.5	.6	.8	.6	-.9	-1.4	-3.1	-6.4	-5.7	18.1
4	.1	.1	.2	.4	.5	.9	.0	-.7	-3.1	-8.4	-6.8	22.4
5	.0	.0	.1	.2	.5	1.0	.5	.3	-3.0	-9.3	-8.1	25.3
6	.0	.0	.1	.1	.3	.9	.7	1.1	-2.2	-9.7	-9.6	28.5
7	.0	.0	.0	.0	.2	.4	.9	1.2	-1.1	-9.5	-9.7	31.0
8	.0	.0	.0	.0	.1	.4	.9	1.3	.1	-9.0	-10.4	33.4
9	.0	.0	.0	.0	.1	.3	.8	1.2	1.2	-8.6	-10.9	35.4
10	.0	.0	.0	.0	.0	.2	.5	1.2	1.8	-7.7	-10.6	37.2
15	.0	.0	.0	.0	.0	.1	.2	.8	2.0	-2.8	-7.3	44.3
20	.0	.0	.0	.0	.0	.0	.0	.4	1.5	1.8	-1.3	49.1
25	.0	.0	.0	.0	.0	.0	.0	.1	.8	3.1	4.6	53.3
30	.0	.0	.0	.0	.0	.0	.0	.1	.4	4.1	9.9	56.7

Percent Deviations of the Asymptotic and Approximate MSFE
from Pooled Monte Carlo Estimates of the Exact MSFE for Maximum Likelihood (n=25)

Table 7a: 100·[1 − (AsyMSFE/PoMSFE)].

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	3.7	4.3	3.6	4.0	3.6	3.8	3.7	4.0	3.8	3.3	2.9	10.9
2	.6	.8	.8	1.7	2.2	3.0	3.3	4.3	4.7	4.9	5.3	19.6
3	.1	.1	.3	.4	1.6	1.9	2.7	3.9	5.5	5.8	7.0	27.1
4	.1	.0	.2	.3	.7	1.3	2.4	3.7	5.7	6.3	8.0	33.1
5	.0	.1	.0	.1	.3	.9	1.6	2.8	4.9	6.6	9.1	37.7
6	.0	.0	.0	.1	.3	.5	1.0	2.3	4.9	7.2	10.4	41.9
7	.0	.0	.0	.1	.2	.4	1.1	2.3	5.2	7.3	11.3	45.8
8	.0	.0	.0	.1	.0	.3	.9	1.7	4.5	7.6	12.1	49.2
9	.0	.0	.0	.0	.0	.1	.6	1.4	4.1	7.4	12.8	52.2
10	.0	.0	.0	.0	.0	.1	.5	1.4	3.7	7.3	13.5	54.9
15	.0	.0	.0	.0	.0	.0	.1	.2	2.2	7.9	16.8	64.5
20	.0	.0	.0	.0	.0	.0	.1	.2	1.5	6.9	19.5	71.0
25	.0	.0	.0	.0	.0	.0	.0	.0	.9	6.0	21.3	75.4
30	.0	.0	.0	.0	.0	.0	.0	.0	.5	5.3	22.1	78.4

Table 7b: 100·[1 − (AppMSFE/PoMSFE)].

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	-.3	.3	-.4	.0	-.4	-.2	-.3	.0	-.2	-.7	-1.1	7.2
2	.6	.6	.2	.3	.0	-.2	-1.0	-.9	-1.5	-2.2	-2.2	13.0
3	.1	.1	.3	.2	.8	.1	-.4	-1.1	-1.5	-3.6	-3.4	18.1
4	.1	.0	.2	.2	.4	.5	.4	-.3	-1.4	-4.8	-5.1	22.3
5	.0	.1	.0	.1	.2	.6	.5	-.2	-1.8	-5.7	-6.1	25.2
6	.0	.0	.0	.1	.3	.4	.5	.1	-1.0	-5.7	-6.7	28.1
7	.0	.0	.0	.1	.2	.3	.8	.9	.2	-5.9	-7.4	31.0
8	.0	.0	.0	.1	.0	.3	.8	.8	.3	-5.5	-7.8	33.4
9	.0	.0	.0	.0	.0	.1	.6	.8	.7	-5.5	-8.1	35.6
10	.0	.0	.0	.0	.0	.1	.5	1.0	1.1	-5.2	-8.3	37.8
15	.0	.0	.0	.0	.0	.0	.1	.2	1.5	-1.1	-6.2	45.3
20	.0	.0	.0	.0	.0	.0	.1	.2	1.4	1.4	-1.9	51.2
25	.0	.0	.0	.0	.0	.0	.0	.0	.8	3.1	2.8	55.4
30	.0	.0	.0	.0	.0	.0	.0	.0	.5	3.8	6.8	58.6

Percent Deviations of the Asymptotic and Approximate MSFE
from Pooled Monte Carlo Estimates of the Exact MSFE for Maximum Likelihood (n=40)

Table 8a: $100 \cdot [1 - (\text{AsyMSFE}/\text{PoMSFE})]$.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	2.2	3.0	2.5	2.9	2.1	2.5	1.9	2.5	2.1	1.9	2.1	9.0
2	.3	.2	.7	.7	1.3	2.2	2.3	2.7	3.0	3.0	3.5	16.5
3	.1	.0	.1	.2	.6	.9	1.5	2.9	3.1	4.5	4.2	22.4
4	.0	.0	.0	.1	.4	.7	1.2	2.4	3.2	5.0	4.8	27.6
5	.0	.0	.0	.1	.2	.5	.8	2.0	3.4	5.3	5.5	32.2
6	.0	.0	.0	.1	.1	.4	.6	1.9	3.4	6.0	5.9	36.5
7	.0	.0	.0	.0	.0	.3	.4	1.4	3.2	6.2	6.7	40.2
8	.0	.0	.0	.0	.1	.2	.4	1.4	2.7	6.3	7.3	43.6
9	.0	.0	.0	.0	.0	.0	.2	1.1	2.4	6.6	8.0	46.5
10	.0	.0	.0	.0	.0	.0	.2	.8	2.0	6.5	8.4	49.1
15	.0	.0	.0	.0	.0	.0	.1	.1	1.3	5.6	10.9	58.2
20	.0	.0	.0	.0	.0	.0	.0	.1	.6	4.6	12.8	64.7
25	.0	.0	.0	.0	.0	.0	.0	.0	.5	3.5	14.0	69.5
30	.0	.0	.0	.0	.0	.0	.0	.0	.2	3.4	14.3	73.2

Table 8b: $100 \cdot [1 - (\text{AppMSFE}/\text{PoMSFE})]$.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	-.4	.5	.0	.4	-.4	.0	-.6	.0	-.4	-.6	-.5	6.7
2	.3	.1	.3	-.1	-.1	.1	-.4	-.6	-.9	-1.5	-1.2	12.2
3	.1	.0	.0	.0	.1	-.1	-.4	-.3	-1.4	-1.3	-2.4	16.6
4	.0	.0	.0	.1	.3	.3	.0	-.1	-1.3	-1.9	-3.5	20.4
5	.0	.0	.0	.1	.1	.3	.2	.1	-.8	-2.3	-4.2	23.9
6	.0	.0	.0	.1	.1	.3	.2	.6	-.3	-2.0	-5.1	27.2
7	.0	.0	.0	.0	.0	.3	.2	.5	.0	-2.0	-5.3	30.1
8	.0	.0	.0	.0	.1	.2	.3	.8	.1	-1.9	-5.6	32.8
9	.0	.0	.0	.0	.0	.0	.2	.7	.3	-1.4	-5.6	35.2
10	.0	.0	.0	.0	.0	.0	.2	.6	.3	-1.3	-5.7	37.2
15	.0	.0	.0	.0	.0	.0	.1	.1	.9	.0	-4.2	44.3
20	.0	.0	.0	.0	.0	.0	.0	.1	.5	1.2	-1.4	49.9
25	.0	.0	.0	.0	.0	.0	.0	.0	.5	1.6	1.5	54.3
30	.0	.0	.0	.0	.0	.0	.0	.0	.2	2.5	4.0	58.0

Before concluding, three issues are worth brief mention: sample size, exogenous variables, and model linearity. The smallest sample size for our numerical results (n=10) is very small in the context of empirical work. However, it may be a reasonable number to use for comparison with empirical work, given that only *one* coefficient is being estimated. The sample size relative to the number of coefficients is a plausible measure in this context (cf. Sargan (1975)); and since much empirical work involves fewer than ten observations per coefficient estimated, n=10 may be *large* rather than small for practical purposes.

Forecasts of endogenous variables often are based on forecasts (rather than known values) of exogenous variables, adding another source of uncertainty. The algebra of Section 2 readily addresses this because it analyzes a complete system: exogenous variables can be included in the system in the same manner as the endogenous variables, but the former are not simultaneously determined with the endogenous variables nor are they Granger-caused by the endogenous variables. However, for even relatively small systems, the AppMSFE in (12) can become awkward to compute because of the large matrices arising from vectorizing and from Kronecker products. Calzolari (1987) provides an ingenious analytical technique which dramatically reduces the computational burden, making the calculation of the AppMSFE feasible for medium- to large-scale models.

The system in Section 2 is linear. Analytic approximations to confidence intervals could be constructed for nonlinear equations (or systems) as well, but Mariano and Brown (1983) show that simulation may be preferable, not only for the MSFE but for the forecast itself. In the context of (9), both the $\mathbf{u}_{n+s-i}$ and the $\hat{\mathbf{A}}$ would be replicated a number of times by Monte Carlo simulation according to their estimated distributions, and Monte Carlo estimates of the forecast mean and the MSFE would be constructed from the resulting "pseudo-forecasts". Marquez (1988) applies this simulation approach to estimate confidence intervals for the response of the US trade account to alternative exchange rate realizations. That analysis examines the sensitivity of the confidence intervals to the two types of uncertainty addressed here. His application also demonstrates that the uncertainty of coefficient estimates can have implications for economic questions other than just those dealing with forecasts, e.g., paths of dynamic multipliers.

5. Conclusions

The Schmidt-Baillie formula provides a simple, accurate analytical approximation to the exact MSFE for the AR(1) process and conveniently summarizes a wealth of computationally intensive calculations given by Orcutt and Winokur (1969) and Hoque, Magnus, and Pesaran (1988) for OLS with and without a constant term, and given herein for maximum likelihood without a constant term. Further, the approximate MSFE can be used to provide confidence intervals for forecasts in instances where formulae for the exact MSFE are not known (e.g., multi-equation, multiple lag, dynamic simultaneous equations systems), and when the exact MSFE may not even exist. In contrast to the asymptotic MSFE, which increases monotonically with the forecast horizon, the exact MSFE can decrease as well as increase as the forecast horizon increases, and the approximate MSFE simply and accurately captures why that occurs. That non-monotonicity can be present even when the economic process has little dynamics. What is important is that the extent of the dynamics is unknown and so must be estimated.[15] For dynamics common to econometric models, the approximate MSFE often has a *maximum* at a forecast horizon of one to twelve periods, i.e., at horizons that are of principal concern to forecasters and policy makers. Although exact results are seldom available for realistic econometric models, the Schmidt-Baillie approximation is easily calculated and appears more accurate than the standard asymptotic formula.

[15]In our framework, autoregressive errors constitute dynamics, even if associated with static models.

Appendix A. The Derivation of $\mathrm{Var}(b_{n,s}|y_n)$.

First, note that $(\mathbf{ABC})^\nu = (\mathbf{A}\otimes\mathbf{C}')\mathbf{B}^\nu$ for conformable matrices **A**, **B**, and **C**, where the Kronecker product $\otimes$ is defined as $(\mathbf{A}\otimes\mathbf{B}) = (b_{ij}\mathbf{A})$. Thus,

$$\text{(A.1)} \qquad \begin{aligned} (\mathbf{S}'\mathbf{A}^s\mathbf{y}_n) &= (\mathbf{I}\mathbf{S}'\mathbf{A}^s\mathbf{y}_n)^\nu \\ &= (\mathbf{I}\otimes\mathbf{y}_n')(\mathbf{S}'\mathbf{A}^s)^\nu \end{aligned}$$

where **I** is the identity matrix. Hence the second term on the RHS of (9) is:

$$\text{(A.2)} \qquad \begin{aligned} \mathbf{S}'(\mathbf{A}^s-\hat{\mathbf{A}}^s)\mathbf{y}_n &= (\mathbf{I}\otimes\mathbf{y}_n')[(\mathbf{S}'\mathbf{A}^s)^\nu-(\mathbf{S}'\hat{\mathbf{A}}^s)^\nu] \\ &= (\mathbf{I}\otimes\mathbf{y}_n')[\mathbf{f}_s(\boldsymbol{\theta}) - \mathbf{f}_s(\hat{\boldsymbol{\theta}})] \end{aligned}$$

where $\mathbf{f}_s(\boldsymbol{\theta}) = (\mathbf{S}'\mathbf{A}^s)^\nu = (\mathbf{S}'\otimes\mathbf{I})(\mathbf{A}^s)^\nu$. The derivative $\partial\mathbf{f}_s(\boldsymbol{\theta})/\partial\boldsymbol{\theta}'$ is $\mathrm{D}(s)'\mathbf{R}$ where:

$$\text{(A.3)} \qquad \begin{aligned} \mathrm{D}(s)' &= \partial(\mathbf{S}'\mathbf{A}^s)^\nu/\partial\boldsymbol{\alpha}' \\ &= (\mathbf{S}'\otimes\mathbf{I})\{\partial(\mathbf{A}^s)^\nu/\partial\boldsymbol{\alpha}'\} \\ &= (\mathbf{S}'\otimes\mathbf{I})\left[\sum_{i=0}^{s-1} \mathbf{A}^i\otimes(\mathbf{A}^{s-i-1})'\right] \end{aligned}$$

by application of the matrix form of the chain rule, noting that:

$$\text{(A.4)} \qquad (\mathbf{A}^s)^\nu = \left[\mathbf{A}^i\mathbf{A}(\mathbf{A}^{s-i-1})\right]^\nu = \left[\mathbf{A}^i\otimes(\mathbf{A}^{s-i-1})'\right](\mathbf{A})^\nu$$

for $i=0,\ldots,s-1$. Expanding $\mathbf{f}_s(\hat{\boldsymbol{\theta}})$ in a Taylor series about $\mathbf{f}_s(\boldsymbol{\theta})$ gives:

$$\text{(A.5)} \qquad \sqrt{n}\cdot[\mathbf{f}_s(\boldsymbol{\theta}) - \mathbf{f}_s(\hat{\boldsymbol{\theta}})] = [\mathrm{D}(s)']\Big|_{\boldsymbol{\theta}^+}\cdot\mathbf{R}\cdot[\sqrt{n}\cdot(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})]$$

where $\boldsymbol{\theta}^+$ lies between $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$. Because $\mathrm{D}(s)'$ is everywhere continuous in $\boldsymbol{\theta}$, then:

$$\text{(A.6)} \qquad \operatorname*{plim}_{n\to\infty} [\mathrm{D}(s)']\Big|_{\boldsymbol{\theta}^+} = [\mathrm{D}(s)']\Big|_{\boldsymbol{\theta}} .$$

By application of Cramér's (1946, p. 299) Linear Transformation Theorem and Mann and Wald's (1943) Corollary 2,

$$\text{(A.7)} \qquad \sqrt{n}\cdot[\mathbf{f}_s(\boldsymbol{\theta}) - \mathbf{f}_s(\hat{\boldsymbol{\theta}})] \xrightarrow{D} \mathrm{N}(0, \mathrm{D}(s)'\Gamma\mathrm{D}(s)) ,$$

and so at last we have:

$$\text{(A.8)} \qquad \mathrm{Var}(\mathbf{b}_{n,s}|\mathbf{y}_n) = n^{-1}\cdot(\mathbf{I}\otimes\mathbf{y}_n')[\mathrm{D}(s)'\Gamma\mathrm{D}(s)](\mathbf{I}\otimes\mathbf{y}_n) ,$$

ignoring terms of $o_p(n^{-1})$.

See Schmidt (1974), Baillie (1979), and Chong and Hendry (1986) for details.

Appendix B. Asymptotic and Approximate MSFE for Hoque, Magnus, and Pesaran (1988) and Orcutt and Winokur (1969)

This appendix lists the asymptotic and approximate MSFE corresponding to the values of (β,n,s) for which the exact MSFE is numerically evaluated in Hoque, Magnus, and Pesaran (1988, Tables 1 and 2) and for which Orcutt and Winokur (1969, Table VII) conducted Monte Carlo experiments.

For HMP, (12′) in the text above is the basis for the calculations of the AsyMSFE and AppMSFE, reported in Table B.1, noting that the AsyMSFE is the first term on the RHS of that equation. The same asymptotic and approximate MSFE are used for both the stationary and non-stationary cases in Tables 1 and 2 in the text.

For Orcutt and Winokur (1969), the AsyMSFE and AppMSFE are derived below, directly from (12), and are reported in Table B.2. However, (3) is not valid for their experiment with β=1.0, so we use β=0.9999 instead. That should offer a good approximation, given the difficulty in finite samples in distinguishing between a unit root and a root close to (but less than) unity.

The derivation of the AsyMSFE and AppMSFE for the AR(1) process with an unknown constant proceeds as follows. Equation (1) is a two-equation system, with the first equation being the AR(1) process and the second equation defining the constant term:

$$\text{(B.1)} \qquad \mathbf{y}_t = \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \mathbf{A}\mathbf{y}_{t-1} + \mathbf{u}_t = \begin{bmatrix} \beta & \alpha \\ 0 & 1 \end{bmatrix} \cdot \mathbf{y}_{t-1} + \begin{bmatrix} u_{1t} \\ 0 \end{bmatrix},$$

where $y_{21} = 1$. The vectors and matrices necessary for solving (12) are as follows.

$$\text{(B.2)} \qquad \boldsymbol{\alpha} \equiv A^{\nu} = (\ \beta\ 0\ \alpha\ 1\ )'$$

$$\boldsymbol{\theta} = (\ \alpha\ \beta\ )'$$

$$\boldsymbol{\Omega} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{S} = (\ 1\ 0\ )'$$

$$\boldsymbol{\Psi} = \begin{bmatrix} \sigma^2+\alpha^2(1+\beta)/(1-\beta) & -\alpha(1+\beta) \\ -\alpha(1+\beta) & (1-\beta^2) \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{r} = (\,0\,0\,0\,1\,)'$$

In Orcutt and Winokur's experiments, α=0, simplifying $\mathbf{\Psi}$ and hence $\mathbf{\Gamma}$ and $\mathbf{D}(s)$.

$$\text{(B.3)} \qquad \mathbf{\Psi} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & (1-\beta^2) \end{bmatrix}$$

$$\mathbf{\Gamma} \equiv \mathbf{R\Psi R}' = \begin{bmatrix} (1-\beta^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}(s)' = \begin{bmatrix} \Sigma\beta^{s-1} & 0 & 0 & 0 \\ 0 & 0 & \Sigma\beta^{j} & 0 \end{bmatrix}$$

The summations in $\mathbf{D}(s)'$ are over j=0,...,s–1. Substituting (B.2)-(B.3) into (12) and simplifying, the AppMSFE for a univariate AR(1) process with an unknown constant is:

$$\text{(B.4)} \qquad \text{AppMSFE} = \sigma^2\left[\frac{1-(\beta^2)^s}{1-\beta^2}\right] + (n^{-1}\cdot\sigma^2)\cdot\left\{(s\beta^{s-1})^2 + \left[\frac{1-\beta^s}{1-\beta}\right]^2\right\},$$

where we have set $y_{in}^2=\sigma^2/(1-\beta^2)$, its unconditional expectation. The first term on the RHS of (B.4) is the AsyMSFE and is the same as when the constant is known. The first term in the braces is the effect from estimating β (as in (12′)); the second term is from estimating α and is additional to what appears in (12′). These three sources of uncertainty can be seen clearly from the generalization of (9′), the equation for the forecast error.

$$\text{(B.5)} \quad (y_{n+s}-\hat{y}_{n+s}) = \sum_{i=0}^{s-1}(\beta^i)\cdot u_{n+s-i} + (\beta^s-\hat{\beta}^s)\cdot y_n + \alpha\sum_{i=0}^{s-1}(\beta^i-\hat{\beta}^i) + (\alpha-\hat{\alpha})\sum_{i=0}^{s-1}\hat{\beta}^i$$

The first two terms on the RHS are the same as in (9′), the third is zero for α=0, and the fourth is the contribution from estimating rather than knowing α. Fuller and Hasza (1980) derived (B.4) directly from (B.5).

Table B.1: Values of AppMSFE and AsyMSFE for HMP (1988, Tables 1–2)

		β											
n	s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
10	1	1.111	1.111	1.111	1.111	1.111	1.111	1.111	1.111	1.111	1.111	1.111	1.111
10	2	1.000	1.014	1.058	1.130	1.231	1.361	1.520	1.708	1.924	2.170	2.304	2.416
10	3	1.000	1.010	1.043	1.106	1.211	1.375	1.619	1.970	2.459	3.122	3.532	3.901
10	4	1.000	1.010	1.042	1.100	1.197	1.356	1.619	2.057	2.778	3.942	4.759	5.556
15	1	1.071	1.071	1.071	1.071	1.071	1.071	1.071	1.071	1.071	1.071	1.071	1.071
15	2	1.000	1.013	1.051	1.116	1.206	1.321	1.463	1.630	1.823	2.041	2.160	2.260
15	3	1.000	1.010	1.043	1.103	1.202	1.353	1.573	1.884	2.313	2.888	3.241	3.558
15	4	1.000	1.010	1.042	1.100	1.194	1.346	1.590	1.982	2.611	3.605	4.292	4.958
20	1	1.053	1.053	1.053	1.053	1.053	1.053	1.053	1.053	1.053	1.053	1.053	1.053
20	2	1.000	1.012	1.048	1.109	1.194	1.303	1.436	1.593	1.775	1.981	2.092	2.186
20	3	1.000	1.010	1.042	1.102	1.198	1.342	1.551	1.844	2.244	2.777	3.103	3.396
20	4	1.000	1.010	1.042	1.099	1.193	1.341	1.576	1.947	2.532	3.445	4.071	4.675
25	1	1.042	1.042	1.042	1.042	1.042	1.042	1.042	1.042	1.042	1.042	1.042	1.042
25	2	1.000	1.012	1.047	1.105	1.187	1.292	1.420	1.572	1.747	1.945	2.053	2.143
25	3	1.000	1.010	1.042	1.101	1.195	1.336	1.538	1.820	2.203	2.712	3.022	3.301
25	4	1.000	1.010	1.042	1.099	1.192	1.339	1.567	1.926	2.487	3.352	3.942	4.510
∞	1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
∞	2	1.000	1.010	1.040	1.090	1.160	1.250	1.360	1.490	1.640	1.810	1.903	1.980
∞	3	1.000	1.010	1.042	1.098	1.186	1.313	1.490	1.730	2.050	2.466	2.717	2.941
∞	4	1.000	1.010	1.042	1.099	1.190	1.328	1.536	1.848	2.312	2.998	3.452	3.882

Table B.2: Values of AppMSFE and AsyMSFE for Orcutt and Winokur (1969, Table VII)

		β				
n	s	0.00	0.30	0.60	0.90	1.00
10	1	1.222	1.222	1.222	1.222	1.222
10	2	1.111	1.318	1.804	2.571	2.889
10	3	1.111	1.321	2.046	3.938	4.999
10	4	1.111	1.323	2.145	5.256	7.553
20	1	1.105	1.105	1.105	1.105	1.105
20	2	1.053	1.198	1.571	2.171	2.421
20	3	1.053	1.204	1.753	3.163	3.946
20	4	1.053	1.205	1.825	4.068	5.682
40	1	1.051	1.051	1.051	1.051	1.051
40	2	1.026	1.143	1.463	1.986	2.205
40	3	1.026	1.150	1.618	2.806	3.461
40	4	1.026	1.151	1.677	3.519	4.819
∞	1	1.000	1.000	1.000	1.000	1.000
∞	2	1.000	1.090	1.360	1.810	2.000
∞	3	1.000	1.098	1.490	2.466	3.000
∞	4	1.000	1.099	1.536	2.998	4.000

Appendix C. The Imprecision of Monte Carlo Estimates of the MSFE

This appendix provides an approximate lower bound on the uncertainty of Monte Carlo estimates of the exact MSFE. This bound is valid for the AR(1) model both with and without estimating a constant term, and it generalizes straightforwardly to the MSFE from a general linear dynamic system by using a Wishart rather than a χ^2 distribution.

In (9), and so in (9′) and (B.5), the source of "inherent" uncertainty is independent from that of coefficient uncertainty. In a Monte Carlo analysis such as Orcutt and Winokur's, all the u_t's are simulated; thus, both the "future" errors u_{n+s-i} and the $\hat{\beta}$ are simulated. Throughout this appendix, we ignore the latter effect because the first dominates, at least for large n. From independence, that results in a lower bound on the associated variability from Monte Carlo simulation. The forecast errors (ignoring coefficient uncertainty) are a linear combination of the future shocks u^{ℓ}_{n+s-i}, where ℓ denotes the ℓ^{th} of L replications.

(C.1) $$(y^{\ell}_{n+s} - \hat{y}^{\ell}_{n+s}) \cong \sum_{i=0}^{s-1} (\beta^i)\cdot u^{\ell}_{n+s-i}$$

The u^{ℓ}_{n+s-i} are jointly normal, so the linear combination of them on the RHS is normal.

(C.2) $$\sum_{i=0}^{s-1} (\beta^i)\cdot u^{\ell}_{n+s-i} \sim N(0, \text{AsyMSFE})$$

The Monte Carlo estimator of the MSFE is the average of the squared forecast errors:

(C.3) $$\text{McMSFE} = \sum_{\ell=1}^{L} (y^{\ell}_{n+s} - \hat{y}^{\ell}_{n+s})^2/L \underset{\text{app}}{\sim} \text{AsyMSFE}\cdot\chi^2(L)/L .$$

Because the first two moments of a $\chi^2(L)$ are L and 2L, and because L typically is quite large, we have the following approximation:

(C.4) $$\text{McMSFE/ExMSFE} \cong \text{McMSFE/AsyMSFE} \underset{\text{app}}{\sim} \chi^2(L)/L \underset{\text{app}}{\sim} N(1, [2/L]) .$$

That is, the estimated MSFE is unbiased for the ExMSFE (which is an exact result, following directly from the estimated MSFE being a sample mean of the ExMSFE), with a percent standard deviation approximately equal to $100\cdot\sqrt{(2/L)}$. Monte Carlo simulation indicates that the approximation errors in (C.4) are small for the values of (β,n,s) in (e.g.) HMP and with $L \geq 100$. Orcutt and Winokur (1969) use L=1000, so the 95% confidence interval on a typical estimate in their study is approximately ±9.0%.

Appendix D. Details of the Monte Carlo Simulation of the MSFE Based on the Maximum Likelihood Estimator

This appendix describes methodological and computational aspects of the Monte Carlo simulation of the MSFE for ML, and tabulates the resulting asymptotic, approximate, and Monte Carlo estimates of the MSFE.

Experimental Design. In the notation and terminology of Hendry (1984), the *data generation process* is (1′) plus (13), the *relationship of interest* is (1′) (but would not be if, e.g., the econometric model were mis-specified), the *objective* of the Monte Carlo study is to estimate the (exact) MSFE over a wide range of the *parameter space* $\Theta \times \mathcal{T}$, so that:

(D.1) $$\theta \equiv (\beta, \sigma^2) \in \Theta = (\theta \mid |\beta|<1; \sigma^2>0)$$

$$\tau \equiv (n, s) \in \mathcal{T} = (\tau \mid n\in[n_a,n_b], s\in[s_a,s_b]) ,$$

where n_a, n_b, s_a, and s_b are pre-assigned. We chose $\sigma^2=1$, and without loss of generality. For β, n, and s, we chose the same values as in Hoque, Magnus, and Pesaran (1988), but included n=40 and s=5,6,...,30 as well, given the interest in the performance of the analytical formulae at medium to long forecast horizons. A full factorial design was adopted for (β,n) given by:

(D.2) $$\beta = (0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99)$$

$$n = (10, 15, 20, 25, 40) ,$$

with 60 experiments in all. For each experiment, the MSFE was estimated at horizons:

(D.3) $$s = (1, 2, 3, 4, 5, 6, ..., 29, 30) .$$

Simulation. For each experiment, L replications of n+s normal pseudo-random numbers $\{(u_t^\ell, t=1,...,n+s), \ell=1,...,L\}$ were generated from pairs of *uniform* pseudo-random numbers using Box and Muller's (1958) transformation.[1] For each replication, a set $(y_t^\ell$, t=1,...,n+s) was created from (1′) and (13) with $\delta^2=(1-\beta^2)^{-1}$ (i.e., stationary y_t's), and the ML estimate was found by solving the cubic in β from setting the score of the likelihood

[1]The uniform random number generators are Carrier, Atkins, and Taylor's (1969) mixed-congruential generator RNDM (but converted from COMPASS to FORTRAN) and NAg's (1984) multiplicative-congruential generator G05CAF. Different random number generators were used for each number in the pair of uniform pseudo-random numbers in order to avoid potential difficulties with Box and Muller's transformation: see Neave (1973).

equal to zero; see Koopmans (1942), Anderson (1971, p. 354), and Beach and MacKinnon (1978). Given the ML estimate, the forecast error was calculated for each value of s. Explicitly, let $\tilde{\beta}=\tilde{\beta}(\beta_h,n_j,\ell)$ denote the ML estimate for the ℓ^{th} replication of the experiment with $\beta=\beta_h$ and $n=n_j$, and let $\{\tilde{g}^{(\ell)}_{n_j+s_k}(\beta_h);\ s_k=1,2,\ldots,30\}$ be the corresponding set of observed forecast errors.

$$\text{(D.4)}\quad \tilde{g}^{(\ell)}_{n_j+s_k}(\beta_h) \equiv y^{(\ell)}_{n_j+s_k}-\tilde{y}^{(\ell)}_{n_j+s_k} = \sum_{i=0}^{s_k-1}(\beta_h^i)\cdot u^{\ell}_{n_j+s_k-i} + (\beta_h^{s_k}-\tilde{\beta}^{s_k})\cdot y^{\ell}_{n_j} .$$

The Monte Carlo estimator of the MSFE is:

$$\text{(D.5)}\qquad \text{McMSFE}_{n_j+s_k}(\beta_h) = \sum_{\ell=1}^{L}[\tilde{g}^{(\ell)}_{n_j+s_k}(\beta_h)]^2/L ,$$

which, in Hendry's (1984) terminology, is the *naive* Monte Carlo estimator. When normalized by the ExMSFE, it is approximately distributed as N(1, [2/L]). In our design, L is 10^4, so the standard deviation of McMSFE/ExMSFE is $\sqrt{(2/10^4)}$ or about 1.4%. Increasing L tenfold would reduce the standard deviation to only about 0.5%, an indication of the difficulties in obtaining precise estimates by such Monte Carlo techniques. Cf. Ansley and Newbold (1980) who compute the McMSFE for several estimators (including ML) of various ARMA processes, but use 1000 or fewer replications per experiment.

Control variates provide a powerful method for variance reduction of naive Monte Carlo estimators; cf. Hammersley and Handscomb (1964) and Hendry (1984) for details. To be useful, a control variate (CV) should be highly correlated with the naive estimator and should have a known distribution.[2] Because the purpose of the Monte Carlo study is to estimate a moment which is *unknown*, those two properties might seem to conflict. However, often it is possible to partition a statistic into an "asymptotic" component and a finite sample one, with the former having an exact distribution; cf. Hendry (1984) on doing so for econometric estimators. The McMSFE has a natural control variate because the first term on the RHS of the equality in (D.4) (and more generally, of (9)) is *exactly* normal, and independent of the second term. The implied control variate is:

---

[2]Often, knowing the first two moments of the CV suffices.

$$\text{(D.6)} \qquad \text{CvMSFE} \;=\; \sum_{\ell=1}^{L} \Big[\sum_{i=0}^{s_k-1} (\beta_h^i) u^{\ell}_{n_j+s_k-i} \Big]^2 / L \;,$$

which, as is shown in Appendix C, is exactly distributed as AsyMSFE$\cdot\chi^2(L)/L$ and has a mean of AsyMSFE.

The CV is used to reduce the simulation uncertainty by subtracting it from the naive estimator (with which it is positively correlated) and adding back the known mean of the CV. The resulting Monte Carlo estimator is called a *pooled* estimator, and here is:

$$\text{(D.7)} \qquad \text{PoMSFE} \;\equiv\; \text{McMSFE} \;-\; \text{CvMSFE} \;+\; E(\text{CvMSFE})$$

$$= \;\text{AsyMSFE} \;+\; \sum_{\ell=1}^{L} [(\beta_h^{s_k} - \tilde{\beta}^{s_k}) \cdot y^{\ell}_{n_j}]^2 / L \;.$$

By construction, the pooled estimator has the same expectation as the naive estimator. Its variance is smaller than that of the naive estimator by the extent to which the CV is correlated with the naive estimator. In the present case, the reduction in variance is obvious because the CV has eliminated the term in the naive estimator which simulates the AsyMSFE. The efficiency of the CV will vary across experiments, but from a cursory comparison of the fluctuations in the naive and pooled estimates, it is readily apparent that they are considerable for the MSFE.

The entire Monte Carlo study for the MSFE with ML took 12 hours 45 minutes on an IBM PS/2 Model 70 (80386 PC running at 20 MHz with an 80387 math chip).

Tables. Tables D.1–D.5 list the values of the PoMSFE for n=10, 15, 20, 25, and 40 respectively. Tables D.6–D.10 list the corresponding values of AppMSFE. Table D.11 lists the values of AsyMSFE (applicable to all values of n).

Table D.1: The Pooled Monte Carlo Estimates
of the Exact MSFE for Maximum Likelihood (PoMSFE) for n=10.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.116	1.106	1.116	1.104	1.106	1.111	1.122	1.091	1.093	1.071	1.067	1.168
2	1.034	1.049	1.081	1.138	1.232	1.331	1.472	1.617	1.782	1.958	2.093	2.631
3	1.011	1.030	1.064	1.135	1.237	1.392	1.606	1.877	2.227	2.694	3.077	4.403
4	1.007	1.022	1.054	1.118	1.227	1.380	1.638	1.990	2.504	3.312	4.019	6.441
5	1.004	1.012	1.050	1.110	1.221	1.378	1.634	2.044	2.693	3.809	4.906	8.788
6	1.001	1.016	1.046	1.104	1.214	1.378	1.622	2.063	2.801	4.245	5.770	11.221
7	1.001	1.014	1.046	1.106	1.209	1.372	1.614	2.071	2.889	4.597	6.602	14.008
8	1.001	1.015	1.043	1.102	1.199	1.364	1.614	2.070	2.921	4.867	7.416	17.080
9	1.001	1.011	1.042	1.102	1.200	1.351	1.599	2.065	2.957	5.092	8.165	20.418
10	1.000	1.012	1.042	1.102	1.201	1.350	1.592	2.048	2.968	5.292	8.833	23.980
11	1.000	1.011	1.042	1.100	1.195	1.349	1.587	2.038	2.981	5.449	9.491	27.659
12	1.000	1.011	1.042	1.100	1.193	1.346	1.581	2.039	2.951	5.593	10.189	31.724
13	1.001	1.010	1.043	1.099	1.191	1.344	1.577	2.026	2.953	5.709	10.819	35.812
14	1.000	1.010	1.042	1.100	1.192	1.342	1.576	2.009	2.952	5.834	11.413	40.078
15	1.000	1.010	1.042	1.100	1.194	1.339	1.576	2.007	2.925	5.858	11.911	44.643
16	1.000	1.010	1.042	1.099	1.193	1.340	1.574	1.994	2.915	5.942	12.413	49.584
17	1.000	1.011	1.043	1.099	1.193	1.337	1.571	1.987	2.905	5.974	12.905	54.764
18	1.000	1.010	1.041	1.099	1.192	1.337	1.568	1.988	2.903	6.032	13.393	59.840
19	1.000	1.011	1.042	1.099	1.191	1.336	1.567	1.985	2.885	6.046	13.799	65.245
20	1.001	1.010	1.042	1.099	1.191	1.335	1.568	1.981	2.870	6.060	14.197	70.630
21	1.000	1.010	1.042	1.099	1.191	1.335	1.567	1.978	2.863	6.093	14.475	75.965
22	1.000	1.010	1.042	1.099	1.191	1.334	1.567	1.979	2.858	6.090	14.783	81.850
23	1.000	1.010	1.042	1.099	1.190	1.333	1.567	1.973	2.852	6.066	15.058	88.023
24	1.000	1.010	1.042	1.099	1.191	1.334	1.565	1.973	2.844	6.054	15.323	94.019
25	1.000	1.010	1.042	1.099	1.191	1.334	1.565	1.974	2.837	6.032	15.574	100.169
26	1.000	1.010	1.042	1.099	1.191	1.334	1.564	1.973	2.829	6.028	15.829	106.487
27	1.000	1.010	1.042	1.099	1.191	1.335	1.564	1.970	2.823	6.024	16.083	112.841
28	1.000	1.010	1.042	1.099	1.191	1.334	1.563	1.971	2.828	5.985	16.388	119.275
29	1.000	1.010	1.042	1.099	1.191	1.334	1.563	1.972	2.833	5.946	16.678	125.746
30	1.000	1.010	1.042	1.099	1.191	1.334	1.563	1.971	2.827	5.925	16.910	131.839

Table D.2: The Pooled Monte Carlo Estimates
of the Exact MSFE for Maximum Likelihood (PoMSFE) for n=15.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.065	1.078	1.083	1.070	1.072	1.067	1.072	1.072	1.062	1.051	1.041	1.152
2	1.014	1.027	1.059	1.127	1.207	1.297	1.447	1.583	1.756	1.941	2.034	2.586
3	1.003	1.016	1.051	1.124	1.220	1.360	1.569	1.839	2.198	2.648	2.977	4.283
4	1.003	1.014	1.046	1.105	1.209	1.369	1.597	1.962	2.477	3.256	3.867	6.280
5	1.000	1.012	1.045	1.104	1.204	1.363	1.607	2.000	2.661	3.756	4.710	8.453
6	1.001	1.010	1.042	1.103	1.198	1.352	1.611	2.032	2.763	4.147	5.491	10.973
7	1.000	1.011	1.042	1.100	1.195	1.348	1.602	2.030	2.820	4.482	6.255	13.634
8	1.000	1.011	1.041	1.100	1.192	1.343	1.590	2.037	2.869	4.737	6.955	16.574
9	1.000	1.010	1.041	1.100	1.193	1.340	1.586	2.027	2.906	4.949	7.618	19.685
10	1.000	1.010	1.042	1.100	1.193	1.342	1.582	2.024	2.917	5.117	8.194	23.056
11	1.000	1.010	1.042	1.100	1.193	1.339	1.580	2.016	2.913	5.267	8.791	26.588
12	1.000	1.010	1.042	1.099	1.192	1.338	1.572	2.003	2.914	5.375	9.358	30.521
13	1.000	1.010	1.042	1.099	1.191	1.335	1.572	1.992	2.905	5.452	9.883	34.334
14	1.000	1.010	1.042	1.099	1.191	1.334	1.569	1.991	2.887	5.515	10.339	38.636
15	1.000	1.010	1.042	1.099	1.191	1.337	1.571	1.985	2.869	5.558	10.714	42.911
16	1.000	1.010	1.042	1.099	1.191	1.335	1.566	1.979	2.862	5.599	11.168	47.296
17	1.000	1.010	1.042	1.099	1.191	1.334	1.567	1.973	2.853	5.641	11.520	51.791
18	1.000	1.010	1.042	1.099	1.190	1.334	1.565	1.972	2.852	5.640	11.864	56.558
19	1.000	1.010	1.042	1.099	1.191	1.334	1.564	1.970	2.844	5.646	12.192	61.286
20	1.000	1.010	1.042	1.099	1.191	1.334	1.564	1.969	2.839	5.641	12.434	66.431
21	1.000	1.010	1.042	1.099	1.191	1.334	1.562	1.969	2.837	5.673	12.763	71.834
22	1.000	1.010	1.042	1.099	1.190	1.335	1.563	1.970	2.831	5.689	13.020	77.132
23	1.000	1.010	1.042	1.099	1.190	1.335	1.563	1.966	2.820	5.688	13.277	82.578
24	1.000	1.010	1.042	1.099	1.191	1.334	1.564	1.962	2.818	5.712	13.518	88.121
25	1.000	1.010	1.042	1.099	1.191	1.334	1.564	1.963	2.811	5.691	13.790	94.025
26	1.000	1.010	1.042	1.099	1.190	1.334	1.564	1.967	2.807	5.712	13.950	99.598
27	1.000	1.010	1.042	1.099	1.191	1.334	1.564	1.964	2.804	5.708	14.167	105.510
28	1.000	1.010	1.042	1.099	1.191	1.333	1.563	1.963	2.799	5.694	14.351	111.464
29	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.964	2.797	5.690	14.520	117.633
30	1.000	1.010	1.042	1.099	1.191	1.333	1.563	1.963	2.791	5.677	14.631	123.824

Table D.3: The Pooled Monte Carlo Estimates of the Exact MSFE for Maximum Likelihood (PoMSFE) for n=20.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.057	1.054	1.054	1.065	1.059	1.059	1.042	1.046	1.047	1.028	1.039	1.123
2	1.008	1.017	1.058	1.109	1.196	1.299	1.413	1.563	1.739	1.896	2.023	2.509
3	1.000	1.013	1.047	1.109	1.207	1.351	1.538	1.819	2.177	2.609	2.937	4.146
4	1.001	1.011	1.043	1.104	1.199	1.353	1.575	1.933	2.456	3.179	3.811	6.024
5	1.000	1.010	1.042	1.101	1.197	1.351	1.583	1.986	2.622	3.653	4.615	8.061
6	1.000	1.010	1.043	1.100	1.194	1.347	1.582	2.009	2.730	4.044	5.336	10.375
7	1.000	1.010	1.042	1.099	1.193	1.340	1.582	2.007	2.803	4.372	6.062	12.865
8	1.000	1.010	1.042	1.099	1.192	1.338	1.579	2.004	2.849	4.640	6.687	15.592
9	1.000	1.010	1.042	1.099	1.191	1.337	1.576	1.996	2.883	4.846	7.265	18.504
10	1.000	1.010	1.042	1.099	1.190	1.336	1.571	1.992	2.891	5.028	7.842	21.550
11	1.000	1.010	1.042	1.099	1.191	1.334	1.568	1.988	2.886	5.170	8.378	24.881
12	1.000	1.010	1.042	1.099	1.191	1.334	1.568	1.983	2.891	5.281	8.841	28.284
13	1.000	1.010	1.042	1.099	1.191	1.334	1.565	1.984	2.876	5.380	9.315	31.905
14	1.000	1.010	1.042	1.099	1.191	1.335	1.567	1.983	2.862	5.457	9.748	35.593
15	1.000	1.010	1.042	1.099	1.190	1.335	1.566	1.977	2.853	5.506	10.133	39.527
16	1.000	1.010	1.042	1.099	1.190	1.334	1.565	1.976	2.840	5.562	10.497	43.559
17	1.000	1.010	1.042	1.099	1.191	1.334	1.567	1.974	2.836	5.607	10.868	47.818
18	1.000	1.010	1.042	1.099	1.191	1.333	1.565	1.971	2.833	5.605	11.204	51.968
19	1.000	1.010	1.042	1.099	1.190	1.333	1.564	1.967	2.829	5.638	11.524	56.220
20	1.000	1.010	1.042	1.099	1.190	1.334	1.563	1.968	2.825	5.672	11.781	60.862
21	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.966	2.824	5.657	12.027	65.446
22	1.000	1.010	1.042	1.099	1.191	1.333	1.563	1.968	2.816	5.642	12.282	70.448
23	1.000	1.010	1.042	1.099	1.191	1.334	1.562	1.965	2.809	5.651	12.491	75.559
24	1.000	1.010	1.042	1.099	1.190	1.334	1.562	1.962	2.804	5.648	12.713	80.547
25	1.000	1.010	1.042	1.099	1.191	1.334	1.563	1.963	2.802	5.621	12.866	85.965
26	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.963	2.797	5.611	13.022	90.991
27	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.963	2.791	5.602	13.183	96.572
28	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.963	2.790	5.596	13.274	102.169
29	1.000	1.010	1.042	1.099	1.190	1.334	1.563	1.963	2.793	5.590	13.428	107.923
30	1.000	1.010	1.042	1.099	1.190	1.334	1.563	1.963	2.788	5.588	13.540	113.626

Table D.4: The Pooled Monte Carlo Estimates
of the Exact MSFE for Maximum Likelihood (PoMSFE) for n=25.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.038	1.045	1.037	1.041	1.038	1.040	1.038	1.042	1.040	1.034	1.030	1.122
2	1.006	1.018	1.049	1.109	1.186	1.289	1.406	1.558	1.721	1.903	2.008	2.463
3	1.001	1.011	1.045	1.103	1.205	1.337	1.532	1.801	2.170	2.618	2.922	4.033
4	1.001	1.010	1.044	1.102	1.198	1.346	1.574	1.920	2.451	3.199	3.750	5.806
5	1.000	1.011	1.042	1.100	1.194	1.344	1.579	1.961	2.608	3.669	4.529	7.713
6	1.000	1.011	1.041	1.100	1.195	1.340	1.576	1.979	2.720	4.068	5.259	9.822
7	1.000	1.010	1.042	1.100	1.192	1.338	1.579	1.994	2.801	4.378	5.922	12.175
8	1.000	1.010	1.042	1.100	1.191	1.338	1.576	1.988	2.827	4.643	6.532	14.692
9	1.000	1.010	1.042	1.099	1.191	1.334	1.572	1.985	2.843	4.831	7.092	17.384
10	1.000	1.010	1.042	1.099	1.191	1.335	1.570	1.986	2.852	4.988	7.603	20.285
11	1.000	1.010	1.042	1.099	1.191	1.333	1.569	1.982	2.863	5.138	8.107	23.305
12	1.000	1.010	1.042	1.099	1.190	1.334	1.567	1.969	2.862	5.250	8.535	26.582
13	1.000	1.010	1.042	1.099	1.190	1.334	1.565	1.964	2.861	5.357	8.960	29.712
14	1.000	1.010	1.042	1.099	1.190	1.334	1.566	1.963	2.848	5.423	9.356	33.256
15	1.000	1.010	1.042	1.099	1.190	1.333	1.564	1.965	2.835	5.471	9.687	36.877
16	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.969	2.840	5.489	10.001	40.741
17	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.968	2.831	5.538	10.296	44.612
18	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.968	2.826	5.543	10.654	48.773
19	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.967	2.821	5.559	10.871	52.859
20	1.000	1.010	1.042	1.099	1.190	1.333	1.564	1.965	2.820	5.569	11.101	57.374
21	1.000	1.010	1.042	1.099	1.191	1.333	1.563	1.964	2.815	5.565	11.277	61.868
22	1.000	1.010	1.042	1.099	1.191	1.333	1.563	1.963	2.817	5.583	11.479	66.301
23	1.000	1.010	1.042	1.099	1.191	1.333	1.562	1.964	2.808	5.593	11.663	71.171
24	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.962	2.808	5.592	11.853	75.694
25	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.960	2.802	5.573	12.025	80.584
26	1.000	1.010	1.042	1.099	1.190	1.334	1.563	1.961	2.802	5.587	12.194	85.633
27	1.000	1.010	1.042	1.099	1.190	1.334	1.563	1.962	2.800	5.568	12.336	90.631
28	1.000	1.010	1.042	1.099	1.190	1.334	1.563	1.961	2.795	5.576	12.412	95.607
29	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.962	2.792	5.554	12.469	100.474
30	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.793	5.545	12.558	105.541

Table D.5: The Pooled Monte Carlo Estimates
of the Exact MSFE for Maximum Likelihood (PoMSFE) for n=40.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.022	1.031	1.025	1.029	1.022	1.026	1.019	1.026	1.022	1.020	1.021	1.099
2	1.003	1.012	1.047	1.098	1.175	1.278	1.392	1.532	1.691	1.866	1.971	2.371
3	1.001	1.010	1.042	1.100	1.193	1.325	1.513	1.781	2.115	2.583	2.836	3.790
4	1.000	1.010	1.042	1.100	1.195	1.338	1.555	1.894	2.388	3.155	3.626	5.359
5	1.000	1.010	1.042	1.100	1.192	1.339	1.566	1.945	2.566	3.619	4.356	7.088
6	1.000	1.010	1.042	1.099	1.191	1.338	1.568	1.971	2.679	4.018	5.009	8.993
7	1.000	1.010	1.042	1.099	1.190	1.337	1.568	1.975	2.743	4.329	5.635	11.023
8	1.000	1.010	1.042	1.099	1.191	1.335	1.568	1.981	2.776	4.575	6.197	13.225
9	1.000	1.010	1.042	1.099	1.190	1.334	1.565	1.979	2.794	4.791	6.721	15.557
10	1.000	1.010	1.042	1.099	1.190	1.334	1.565	1.975	2.801	4.942	7.185	17.967
11	1.000	1.010	1.042	1.099	1.191	1.333	1.565	1.970	2.809	5.051	7.630	20.450
12	1.000	1.010	1.042	1.099	1.190	1.333	1.564	1.966	2.812	5.142	8.039	23.063
13	1.000	1.010	1.042	1.099	1.190	1.333	1.564	1.969	2.809	5.240	8.417	25.659
14	1.000	1.010	1.042	1.099	1.190	1.333	1.564	1.964	2.808	5.303	8.742	28.448
15	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.963	2.810	5.340	9.045	31.282
16	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.963	2.808	5.368	9.335	34.305
17	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.964	2.806	5.382	9.601	37.339
18	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.965	2.804	5.407	9.821	40.631
19	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.963	2.798	5.418	10.034	43.852
20	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.963	2.794	5.435	10.252	47.145
21	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.962	2.792	5.457	10.415	50.553
22	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.961	2.794	5.457	10.596	54.235
23	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.962	2.794	5.432	10.730	57.833
24	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.961	2.790	5.436	10.869	61.580
25	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.962	2.791	5.427	11.002	65.137
26	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.789	5.426	11.126	68.910
27	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.786	5.416	11.210	72.868
28	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.785	5.436	11.307	76.749
29	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.784	5.445	11.298	80.818
30	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.961	2.783	5.440	11.420	84.840

Table D.6: The Approximate MSFE (AppMSFE) for n=10.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.111	1.111	1.111	1.111	1.111	1.111	1.111	1.111	1.111	1.111	1.111	1.111
2	1.000	1.014	1.058	1.130	1.231	1.361	1.520	1.708	1.924	2.170	2.304	2.416
3	1.000	1.010	1.043	1.106	1.211	1.375	1.619	1.970	2.459	3.122	3.532	3.901
4	1.000	1.010	1.042	1.100	1.197	1.356	1.619	2.057	2.778	3.942	4.759	5.556
5	1.000	1.010	1.042	1.099	1.192	1.343	1.600	2.066	2.946	4.624	5.958	7.368
6	1.000	1.010	1.042	1.099	1.191	1.337	1.583	2.047	3.016	5.171	7.109	9.327
7	1.000	1.010	1.042	1.099	1.191	1.335	1.573	2.023	3.030	5.597	8.197	11.422
8	1.000	1.010	1.042	1.099	1.190	1.334	1.568	2.002	3.012	5.915	9.210	13.642
9	1.000	1.010	1.042	1.099	1.190	1.333	1.565	1.988	2.981	6.141	10.144	15.979
10	1.000	1.010	1.042	1.099	1.190	1.333	1.564	1.977	2.946	6.291	10.993	18.423
11	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.971	2.912	6.379	11.758	20.965
12	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.967	2.883	6.419	12.438	23.596
13	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.964	2.858	6.421	13.037	26.309
14	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.963	2.838	6.395	13.556	29.095
15	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.962	2.823	6.348	14.001	31.948
16	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.811	6.288	14.375	34.860
17	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.802	6.219	14.684	37.825
18	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.795	6.146	14.932	40.836
19	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.790	6.071	15.125	43.886
20	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.787	5.996	15.267	46.970
21	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.784	5.924	15.364	50.083
22	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.782	5.856	15.420	53.219
23	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.781	5.792	15.440	56.373
24	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.780	5.732	15.428	59.540
25	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.678	15.388	62.716
26	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.628	15.324	65.897
27	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.583	15.238	69.077
28	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.543	15.136	72.254
29	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.507	15.018	75.424
30	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.476	14.889	78.583

Table D.7: The Approximate MSFE (AppMSFE) for n=15.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.071	1.071	1.071	1.071	1.071	1.071	1.071	1.071	1.071	1.071	1.071	1.071
2	1.000	1.013	1.051	1.116	1.206	1.321	1.463	1.630	1.823	2.041	2.160	2.260
3	1.000	1.010	1.043	1.103	1.202	1.353	1.573	1.884	2.313	2.888	3.241	3.558
4	1.000	1.010	1.042	1.100	1.194	1.346	1.590	1.982	2.611	3.605	4.292	4.958
5	1.000	1.010	1.042	1.099	1.192	1.339	1.583	2.008	2.779	4.197	5.300	6.453
6	1.000	1.010	1.042	1.099	1.191	1.336	1.575	2.006	2.863	4.673	6.254	8.035
7	1.000	1.010	1.042	1.099	1.191	1.334	1.569	1.996	2.896	5.048	7.146	9.698
8	1.000	1.010	1.042	1.099	1.190	1.334	1.566	1.985	2.901	5.334	7.972	11.436
9	1.000	1.010	1.042	1.099	1.190	1.333	1.564	1.977	2.891	5.545	8.729	13.242
10	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.971	2.874	5.695	9.417	15.111
11	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.967	2.857	5.796	10.036	17.037
12	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.964	2.841	5.856	10.589	19.015
13	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.963	2.826	5.886	11.078	21.040
14	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.962	2.815	5.892	11.506	23.106
15	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.805	5.881	11.877	25.210
16	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.798	5.858	12.194	27.346
17	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.793	5.826	12.462	29.511
18	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.789	5.788	12.684	31.700
19	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.786	5.748	12.864	33.910
20	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.783	5.707	13.007	36.136
21	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.782	5.666	13.115	38.376
22	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.781	5.626	13.192	40.626
23	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.780	5.588	13.243	42.883
24	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.553	13.269	45.144
25	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.520	13.273	47.407
26	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.490	13.260	49.667
27	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.463	13.230	51.924
28	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.438	13.186	54.173
29	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.416	13.131	56.414
30	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.396	13.065	58.645

Table D.8: The Approximate MSFE (AppMSFE) for n=20.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.053	1.053	1.053	1.053	1.053	1.053	1.053	1.053	1.053	1.053	1.053	1.053
2	1.000	1.012	1.048	1.109	1.194	1.303	1.436	1.593	1.775	1.981	2.092	2.186
3	1.000	1.010	1.042	1.102	1.198	1.342	1.551	1.844	2.244	2.777	3.103	3.396
4	1.000	1.010	1.042	1.099	1.193	1.341	1.576	1.947	2.532	3.445	4.071	4.675
5	1.000	1.010	1.042	1.099	1.191	1.337	1.575	1.981	2.700	3.994	4.988	6.019
6	1.000	1.010	1.042	1.099	1.191	1.335	1.571	1.987	2.790	4.437	5.849	7.423
7	1.000	1.010	1.042	1.099	1.191	1.334	1.567	1.983	2.833	4.787	6.648	8.882
8	1.000	1.010	1.042	1.099	1.190	1.334	1.565	1.977	2.848	5.058	7.385	10.391
9	1.000	1.010	1.042	1.099	1.190	1.333	1.564	1.972	2.848	5.263	8.059	11.946
10	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.968	2.841	5.413	8.670	13.543
11	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.965	2.831	5.519	9.221	15.177
12	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.963	2.821	5.590	9.714	16.845
13	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.962	2.811	5.633	10.151	18.544
14	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.962	2.804	5.654	10.536	20.269
15	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.797	5.660	10.871	22.018
16	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.792	5.654	11.162	23.787
17	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.788	5.639	11.410	25.572
18	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.786	5.619	11.619	27.372
19	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.783	5.595	11.794	29.184
20	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.782	5.570	11.936	31.004
21	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.781	5.543	12.050	32.831
22	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.780	5.517	12.137	34.661
23	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.492	12.202	36.493
24	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.468	12.246	38.325
25	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.445	12.272	40.155
26	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.425	12.282	41.979
27	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.406	12.278	43.798
28	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.388	12.262	45.609
29	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.373	12.236	47.410
30	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.359	12.202	49.200

Table D.9: The Approximate MSFE (AppMSFE) for n=25.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.042	1.042	1.042	1.042	1.042	1.042	1.042	1.042	1.042	1.042	1.042	1.042
2	1.000	1.012	1.047	1.105	1.187	1.292	1.420	1.572	1.747	1.945	2.053	2.143
3	1.000	1.010	1.042	1.101	1.195	1.336	1.538	1.820	2.203	2.712	3.022	3.301
4	1.000	1.010	1.042	1.099	1.192	1.339	1.567	1.926	2.487	3.352	3.942	4.510
5	1.000	1.010	1.042	1.099	1.191	1.336	1.571	1.965	2.654	3.876	4.807	5.766
6	1.000	1.010	1.042	1.099	1.191	1.334	1.568	1.976	2.748	4.300	5.612	7.066
7	1.000	1.010	1.042	1.099	1.191	1.334	1.566	1.976	2.796	4.636	6.358	8.405
8	1.000	1.010	1.042	1.099	1.190	1.333	1.564	1.972	2.817	4.898	7.043	9.781
9	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.969	2.823	5.099	7.668	11.190
10	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.966	2.821	5.249	8.235	12.628
11	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.964	2.815	5.358	8.745	14.092
12	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.963	2.809	5.434	9.203	15.580
13	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.962	2.803	5.485	9.610	17.088
14	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.797	5.515	9.969	18.614
15	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.792	5.531	10.285	20.156
16	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.789	5.535	10.559	21.710
17	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.786	5.530	10.796	23.275
18	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.784	5.520	10.998	24.848
19	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.782	5.506	11.169	26.427
20	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.781	5.489	11.312	28.011
21	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.780	5.472	11.428	29.596
22	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.454	11.522	31.182
23	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.436	11.595	32.766
24	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.418	11.649	34.347
25	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.402	11.687	35.924
26	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.386	11.711	37.495
27	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.372	11.723	39.058
28	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.359	11.724	40.613
29	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.347	11.715	42.157
30	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.337	11.698	43.691

Table D.10: The Approximate MSFE (AppMSFE) for n=40.

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.026	1.026	1.026	1.026	1.026	1.026	1.026	1.026	1.026	1.026	1.026	1.026
2	1.000	1.011	1.044	1.099	1.176	1.276	1.397	1.540	1.706	1.893	1.995	2.081
3	1.000	1.010	1.042	1.100	1.192	1.327	1.520	1.786	2.144	2.618	2.905	3.162
4	1.000	1.010	1.042	1.099	1.191	1.335	1.555	1.896	2.419	3.216	3.754	4.268
5	1.000	1.010	1.042	1.099	1.191	1.335	1.564	1.942	2.587	3.704	4.541	5.396
6	1.000	1.010	1.042	1.099	1.191	1.334	1.565	1.960	2.686	4.099	5.267	6.544
7	1.000	1.010	1.042	1.099	1.190	1.334	1.564	1.965	2.742	4.414	5.934	7.709
8	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.965	2.772	4.663	6.543	8.890
9	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.964	2.786	4.858	7.097	10.084
10	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.963	2.792	5.008	7.598	11.290
11	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.962	2.793	5.122	8.050	12.506
12	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.962	2.792	5.207	8.456	13.730
13	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.790	5.269	8.819	14.960
14	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.788	5.312	9.142	16.196
15	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.785	5.342	9.427	17.435
16	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.784	5.361	9.679	18.676
17	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.782	5.371	9.899	19.917
18	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.781	5.376	10.091	21.159
19	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.780	5.376	10.256	22.398
20	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.780	5.373	10.399	23.635
21	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.367	10.520	24.868
22	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.779	5.361	10.622	26.096
23	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.353	10.707	27.318
24	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.346	10.777	28.534
25	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.338	10.834	29.741
26	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.331	10.878	30.941
27	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.323	10.912	32.131
28	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.317	10.936	33.311
29	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.311	10.953	34.481
30	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.305	10.962	35.639

Table D.11: The Asymptotic MSFE (AsyMSFE).

	β											
s	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.000	1.010	1.040	1.090	1.160	1.250	1.360	1.490	1.640	1.810	1.902	1.980
3	1.000	1.010	1.042	1.098	1.186	1.313	1.490	1.730	2.050	2.466	2.717	2.941
4	1.000	1.010	1.042	1.099	1.190	1.328	1.536	1.848	2.312	2.998	3.452	3.882
5	1.000	1.010	1.042	1.099	1.190	1.332	1.553	1.905	2.480	3.428	4.116	4.805
6	1.000	1.010	1.042	1.099	1.190	1.333	1.559	1.934	2.587	3.777	4.714	5.709
7	1.000	1.010	1.042	1.099	1.190	1.333	1.561	1.947	2.656	4.059	5.255	6.596
8	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.954	2.700	4.288	5.742	7.464
9	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.958	2.728	4.473	6.182	8.316
10	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.959	2.746	4.623	6.580	9.150
11	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.960	2.757	4.745	6.938	9.968
12	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.960	2.765	4.843	7.262	10.770
13	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.961	2.769	4.923	7.554	11.556
14	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.961	2.772	4.988	7.817	12.326
15	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.961	2.774	5.040	8.055	13.080
16	1.000	1.010	1.042	1.099	1.190	1.333	1.562	1.961	2.776	5.082	8.270	13.820
17	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.776	5.117	8.463	14.545
18	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.777	5.145	8.638	15.256
19	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.777	5.167	8.796	15.952
20	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.777	5.185	8.938	16.635
21	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.200	9.067	17.304
22	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.212	9.183	17.959
23	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.222	9.288	18.602
24	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.230	9.382	19.232
25	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.236	9.467	19.849
26	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.241	9.544	20.454
27	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.245	9.614	21.047
28	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.249	9.676	21.628
29	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.251	9.733	22.198
30	1.000	1.010	1.042	1.099	1.190	1.333	1.563	1.961	2.778	5.254	9.784	22.756

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