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A Note on Government Gold Policies*

by

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Government gold policies are under active discussion. Recently there have been significant sales of gold by Belgium and the Netherlands, proposed future sales by Switzerland, and rumors of additional sales. This note is an analysis of several government gold policies, including the immediate sale of government gold stocks.

In Chart 1, the left-hand pie chart shows that governments own about one-fifth of the estimated total world gold stock of about 5900 million troy ounces, which is the sum of government stocks and estimates of private aboveground stocks and gold yet to be mined. The right-hand pie chart shows that the United States owns about one quarter of government gold stocks of about 1100 million ounces.

As shown in the middle panel, governments have been net sellers of gold over the period 1974-96. Cumulated net sales, the last number in the last column, have been 72 million ounces. Most recently, in the 1989-96 period, total net sales were 64 million ounces, including the sale by the Netherlands of about 10 million ounces, one third of its holdings, in late 1996.

The bottom panel shows the real gold price, that is the dollar price deflated by the U.S. CPI, over the period since 1968 when governments ended their defense of the official dollar price. Over the period as a whole, the price has varied widely. Since 1992, however, it has fluctuated within a range of about $50. Actual, proposed, and rumored government sales have no doubt put downward pressure on the price during this later period, especially over the last year.

As shown in the top panel of Chart 2, gold has both government uses and private uses. Governments use gold as a monetary asset, as part of a "war chest", and as a strategic material. Private uses can be divided into two categories: depletion uses that reduce the stock and service uses that do not. Depletion uses include electronics, other industrial uses, and dentistry. Service uses include jewelry, bars, coins, and medals.

The bottom panel lists two important considerations that underlie the analysis in this note. First, total economic welfare increases if making government gold available to private agents raises welfare from private uses by more than it reduces welfare from government uses. Second, each government makes more revenue if it sells its gold before other governments either sell or announce a sale. Thus, without coordination there could be a rush to sell, which could strain relations among countries and cause abrupt changes in the gold market.

In this note, we focus on the effects of several government gold policies on the gold market and on welfare from private uses. The top panel of Chart 3 lists two principles for maximizing welfare from private uses. The first principle is that when a resource can be obtained from one stock with no extraction cost, costly extraction from other stocks should be delayed. Violation of this principle leads to a "production inefficiency." If governments withhold their gold for a time, gold is made available from the mines by incurring sizeable costs of extraction instead of from government stocks with no costs of extraction. There is a production inefficiency unless extraction is costless. The second principle listed in the panel is that a resource that can generate welfare should not be withheld from users. Violation of this principle leads to a "use inefficiency." If governments withhold their gold for a time, private uses of gold are too low now and too high later. There is a use inefficiency even if extraction is costless.

It is important to get a sense for the orders of magnitude of the effects of different government gold policies. In order to do so, we use a simulation model described in detail in Henderson, Salant, Irons, and Thomas (1997). The model includes the three key relationships listed in the middle panel of Chart 3. The first relationship is that gold will be mined both today and tomorrow only if net revenue, that is the gold price minus the cost of extraction, is positive and only if net revenue from extraction today is equal to the discounted net revenue from extraction tomorrow. Users may obtain gold by outright purchase or through a gold loan. A gold loan involves receiving gold today and returning the same amount of gold and a loan fee at some future date. The second relationship is that gold will be held both today and tomorrow only if today's price is equal to the discounted sum of tomorrow's price and the loan fee or, equivalently, only if the sum of the price increase and the loan fee expressed as percentages of today's price is equal to the interest rate. The third relationship is that the initial price must be set so that the sum of depletion uses from now on equals the total available stock, including both abovegound and belowground gold.

The numerical assumptions used to calibrate the model are listed in the bottom panel of Chart 3. The constant real cost of extraction of $300 per ounce is an approximation based on industry estimates. The estimate of the one-year real interest rate, 2.5 percent, is a common one. Depletion demand and service demand depend on the price, the loan fee, and population. Population is projected to level off at twice its current value by about 2050. The constant terms and elasticities in the demand equations are chosen so that initial depletion demand equals an average of depletion demand in recent years; the initial real price equals $350, a value close to the current price; and initial service demand equals the current estimated private aboveground stock.

Chart 4 summarizes predictions of the impact on the gold market of two extreme government gold policies: no sale of government gold, the solid lines, and an immediate sale of all government gold, the dotted lines. The top left panel shows that an immediate sale causes the price to drop at once from $350 to about $309 per ounce and to remain below the no sale path thereafter. The top right panel shows that with an immediate sale the service stock--that is, the gold in jewelry, bars, coins, and medals--is higher initially and in most periods and is never lower.

As noted earlier, the postponement of costly mining is one source of the increase in welfare from private uses that is achieved by making government gold available. The middle left panel shows that with no sale, mining continues to occur and falls slowly until 2029 when the mines are projected to be exhausted. By contrast, with an immediate sale, the mines shut down at once, reopen again in the year 2008 and are exhausted in 2056. The reopening and exhaustion of the mines are predicted to be abrupt only because of the approximation of a constant unit cost of extraction. It is profitable to postpone mining for several periods after an immediate sale because in each of those periods tomorrow's price is high relative to today's. Tomorrow's price must be high relative to today's in order to induce private aboveground stock owners to hold gold. This inducement is necessary because the loan fee must be lower given that the service stock is higher. The middle right panel shows that with an immediate sale depletion uses are higher in every period because the price is lower.

The bottom panel of Chart 4 shows the estimated effects on welfare from private uses of the sale of the total government gold stock at different times. These effects are measured in terms of economic surplus (consumer surplus and producer surplus). The first column shows how welfare changes with an immediate sale versus no sale. Total welfare increases by $368 billion because the production and use inefficiencies are eliminated. Most of the increase takes the form of government revenue in the first instance. Depletion users and service users gain, but private aboveground stock owners and mine owners lose. The second column shows how welfare changes with a sale twenty years from now versus no sale. The pattern of gains and losses is similar, but the magnitudes are somewhat different. Some may find it implausible that governments would never sell their gold, so in the third column we present the welfare effects of an immediate sale versus a sale in 2017. Total welfare is $130 billion higher with an immediate sale because the production and use inefficiencies are eliminated at once. An important result not shown in the chart is that a large share of the welfare gain, about 37 percent, comes from eliminating the production inefficiency.

The top panel of Chart 5 shows why government revenue is higher with an immediate sale versus a sale in 2017. With an immediate sale, the dotted line, the price falls to about $309, then increases at a rate less than the rate of interest, and reaches $332 by 2017. It increases at a rate less than the rate of interest because the return to holding gold includes not only price appreciation but also the loan fee. If governments invest their revenue, the dot/dash line, it grows at the real rate of interest of 2.5 percent and reaches about $506 per ounce in 2017, a level considerably above the gold price at that time, $332. If governments do not sell until 2017, the solid line, the price is higher over the next 20 years; as a consequence, depletion is smaller. Therefore, in 2017, after a sale, the total stock is larger, and the price, at $317 per ounce, is lower, than they would be with an immediate sale. It follows that with an immediate sale, government revenue is about $189 per ounce higher in 2017, as indicated by the gap between the dot-dash line and the solid line.

Governments can achieve a welfare gain roughly equal to that from an immediate sale through alternative policies. One such policy is specified in the bottom panel of Chart 5. Under this alternative policy, governments loan out all their remaining gold in each period. In the future when all gold now owned by private agents, whether above or below ground, has been used up, governments sell in every period whatever gold is necessary to make the price be what it would have been if they had sold all their gold immediately. The quantities of gold available for private uses are the same under the alternative policy as with an immediate sale. However, there is an important difference: under the alternative policy, governments relinquish title to their gold in the future and then only gradually. Therefore, to the extent that government uses can be satisfied by owning gold but not physically possessing it, most if not all of the gains associated with maximizing welfare from private uses can be obtained with little or no reduction in welfare from government uses until sometime in the future.

Up to this point, we have considered actions that might be taken by all governments acting together. Of course, one government may sell even if others do not. As shown in Chart 6, if the United States sells all its gold but other governments do not, the price is estimated to drop only to about $340. U.S. receipts are about $89 billion, about 10 percent higher than if all governments sold. A credible announcement by other governments that they intend to sell gold soon has almost the same effect as an immediate sale. Thus, the U.S. example illustrates the consideration that each government makes more revenue if it sells before other governments either sell or announce a sale. This consideration may be important in explaining why some governments have made sizeable sales over the last several years and why there are rumors of future sales.

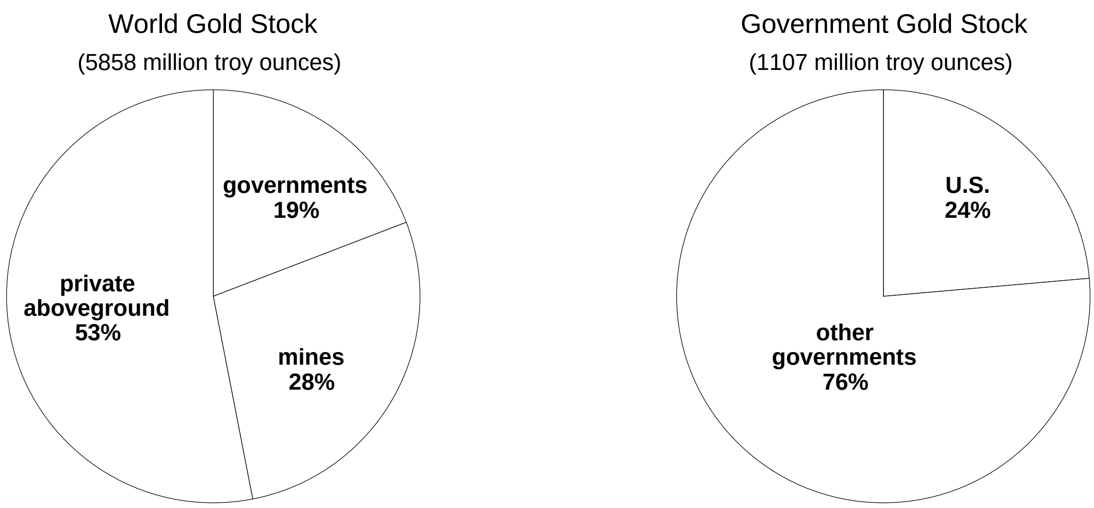
The estimate of the price drop caused by a U.S. sale reported in Chart 6 is based on the assumption that expected sales by other governments remain unchanged. One reason why the actual price drop might be larger is that a U.S. sale might cause an increase in expected sales by other governments.

Reference

Henderson, Dale, Stephen Salant, John Irons, and Sebastian Thomas, (1997), "Can Official Gold Be Put to Better Use?: Qualitative and Quantitative Effects of Alternative Policies"

Chart 1

Gold Stocks



Net Sales of Government Gold
(millions of troy ounces)

Year	IMF	United States	Canada	Belgium	Netherlands	Other *	Net Sales*	Cumulative Net Sales
1974-79	21	17				2	40	40
1980-88	2		4			(38)	(32)	8
1989-96			12	21	23	8	**64**	**72**
memo:1996				7	**10**	(9)	7	

*numbers in ( ) are purchases

Real Gold Price

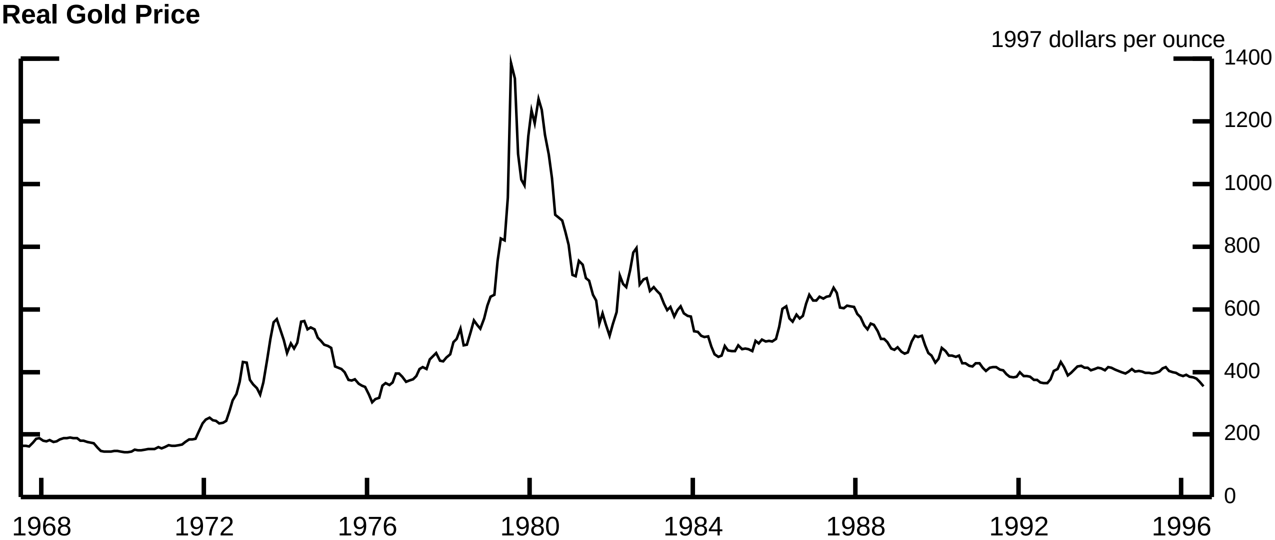


Chart 2

Uses of Gold

Government Uses

- monetary asset
- part of a "war chest"
- strategic material

Private Uses

- **Depletion Uses**
  - electronics
  - other industrial uses
  - dentistry
- **Service Uses**
  - jewelry
  - bars, coins, medals

Two Important Considerations

- Total economic welfare increases if making government gold available to private agents raises welfare from private uses by more than it reduces welfare from government uses.

- Each government makes more revenue if it sells its gold before other governments either sell or announce a sale.

Two Principles for Maximizing Welfare from Private Uses

- When a resource can be obtained from one stock with no extraction cost, costly extraction from other stocks should be delayed.

  Violation leads to a "production inefficiency."

- A resource that can generate welfare should not be withheld from users.

  Violation leads to a "use inefficiency."

Key Relationships

1. Gold will be mined both today and tomorrow only if

$$(\textit{Net Revenue})_t = (\textit{Price})_t - (\textit{Cost of Extraction}) > 0$$

and

$$(\textit{Net Revenue})_t = \frac{(\textit{Net Revenue})_{t+1}}{(1 + \textit{Interest Rate})}$$

2. Gold will be held both today and tomorrow only if

$$(\textit{Price})_t = \frac{(\textit{Price})_{t+1} + (\textit{Loan Fee})_{t+1}}{(1 + \textit{Interest Rate})}$$

or

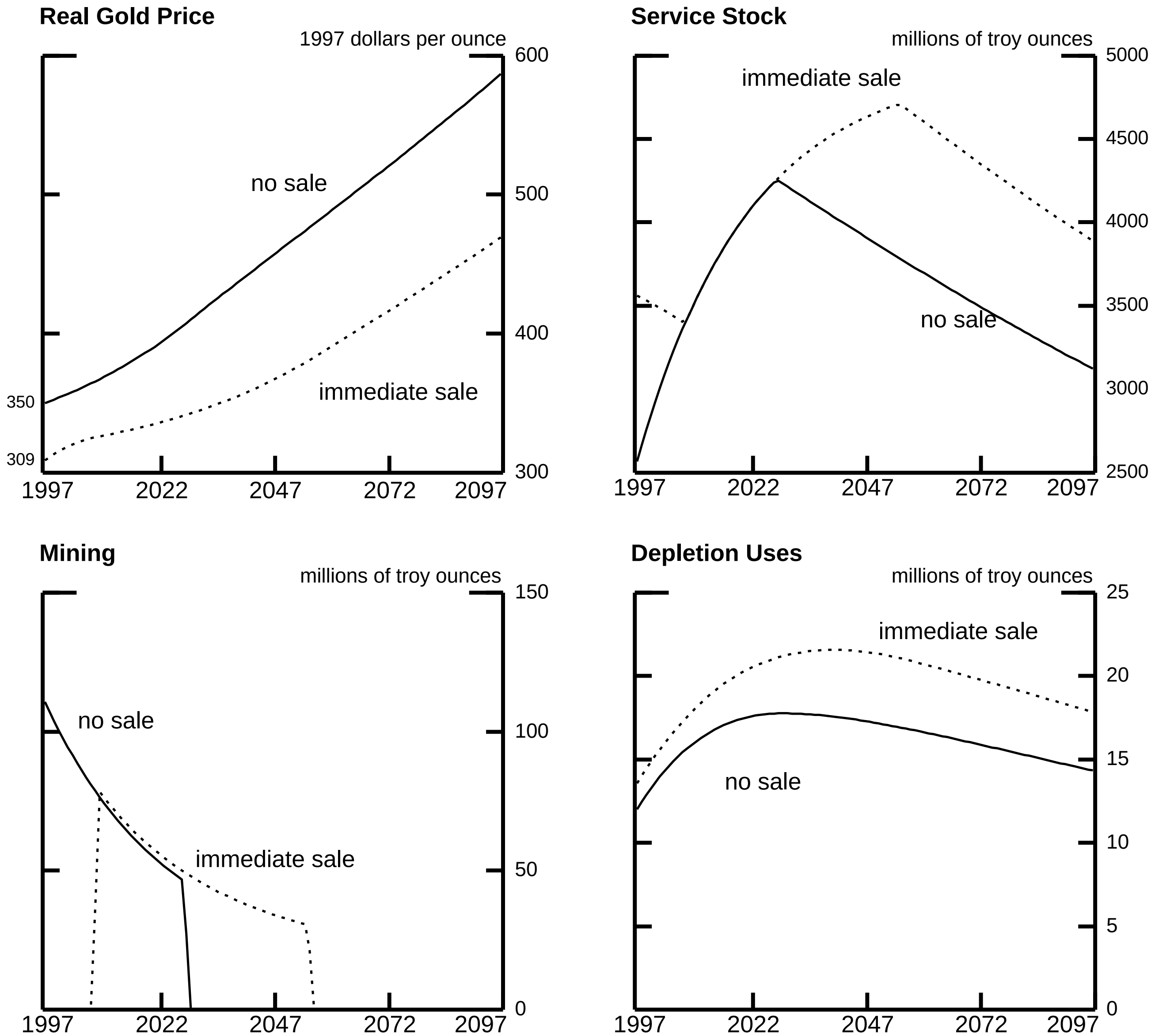
$$\frac{(\textit{Price})_{t+1} - (\textit{Price})_t}{(\textit{Price})_t} + \frac{(\textit{Loan Fee})_{t+1}}{(\textit{Price})_t} = \textit{Interest Rate}$$

3. Initial price set so that sum of depletion uses from now on equals total available stock.

Numerical Assumptions

Cost of extraction	\$300 per ounce
Real interest rate	2.5%
Depletion demand	$(\text{price})^{-.98}$ x population x constant
Service demand	$(\text{loan fee})^{-.98}$ x population x constant
Population index	$2 - (.96)^{t-1}$ starts at one and levels off at two by about 2050

Chart 4

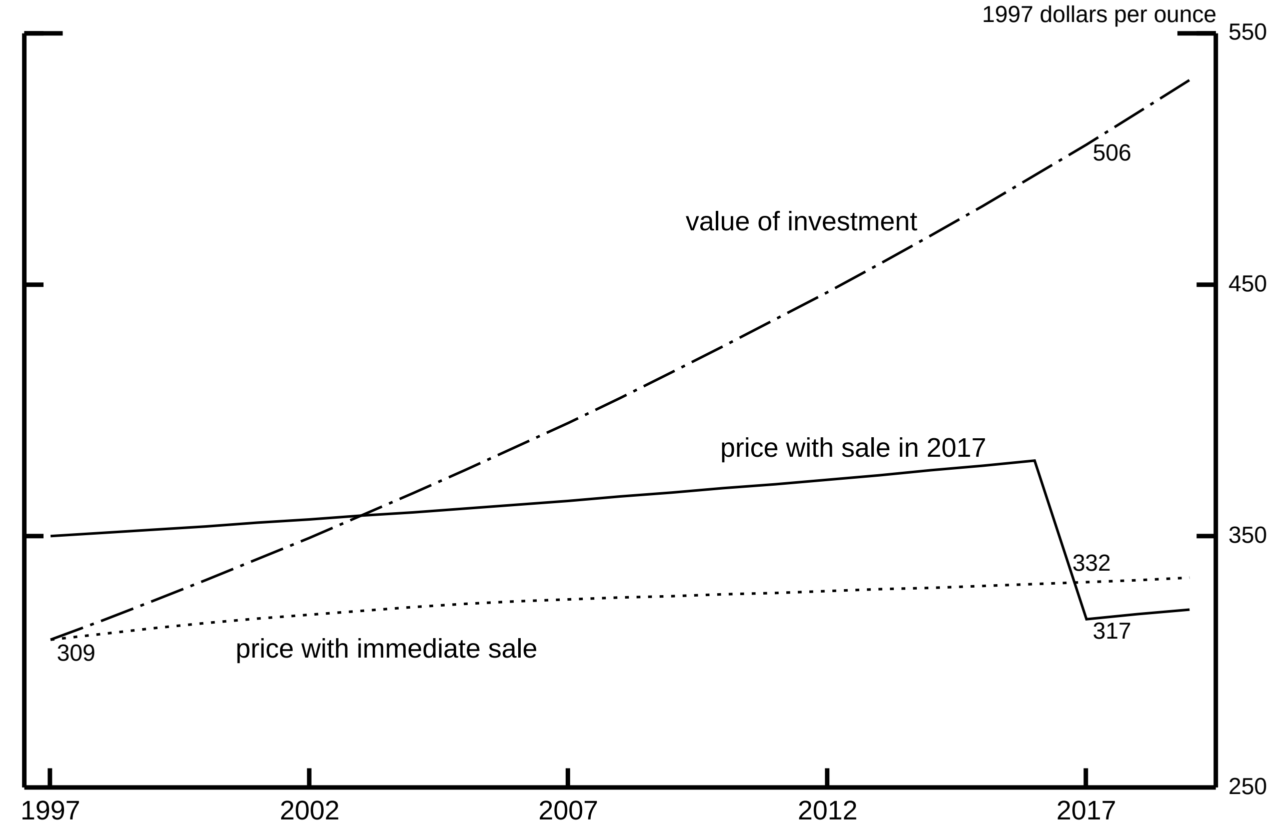


Estimated Effects on Welfare from Private Uses
(billions of 1997 dollars)

	(1) Immediate sale vs. No sale	(2) Sale in 2017 vs. No sale	(1) - (2) Immediate sale vs. Sale in 2017
Total	**368**	**238**	**130**
Government Revenue	342	214	128
Depletion Users	49	41	8
Service Users	149	155	-6
Private Aboveground Stock Owners	-102	-153	51
Mine Owners	-70	-19	-51

Chart 5

Government Revenue with Immediate Sale vs. Sale in 2017

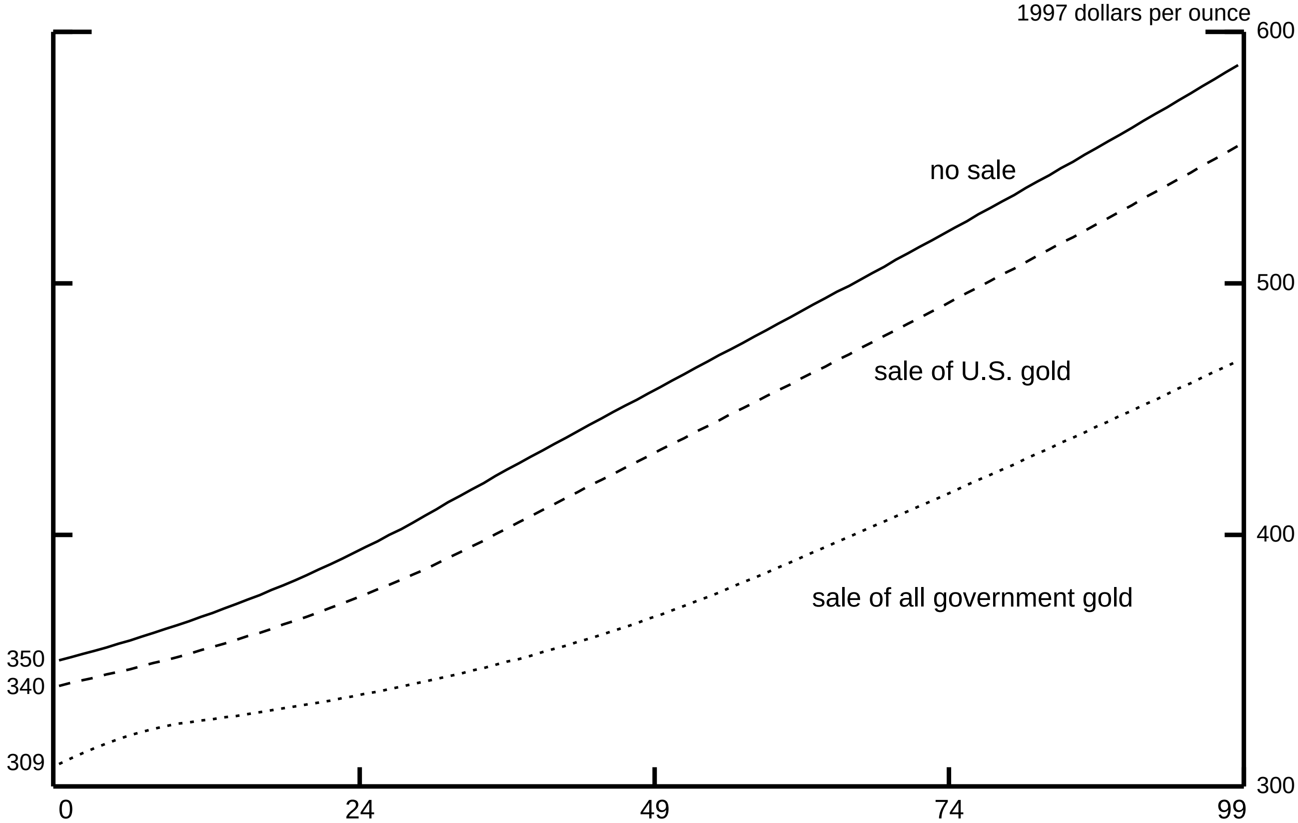


Alternative Policy with Same Private Uses As Immediate Sale

- Governments loan out all their remaining gold in each period.
- When all gold now owned by private agents has been used up, governments sell whatever gold necessary to make price what it would have been if they had sold all their gold immediately.
- Quantities of gold available for private uses same as with immediate sale.
- Main difference is governments relinquish title to gold in future and then only gradually.
- Most of gains associated with maximizing welfare from private uses can be obtained with little or no reduction in welfare from government uses until sometime in future.

Chart 6

Effect on Predicted Path of Real Gold Price from Government Gold Sales



Supplementary Charts

(charts not mentioned in note)

Chart A

Gold Stocks
(end 1995)

	Million Troy Ounces	Percent
Government[1]	**1107**	**19**
United States	262	4
Other industrial countries, EMI, BIS	585	10
Developing countries	157	3
IMF	103	2
Private Aboveground Stock	**2468**	**42**
Jewelry	1862	32
Bars, coins, medals	607	10
Mines	**2283**	**39**
United States	196	3
South Africa	1190	20
Former Soviet Union	215	4
Other	682	12
Total	**5858**	**100**

1. End November 1996

Chart B

Depletion Uses, Increases in Service Stock, and Sources
(millions of troy ounces)

	1993	1994	1995	Average 1993 - 95
Depletion Uses	**11**	**12**	**12**	**12**
Electronics	6	6	7	6
Other industrial uses	3	3	4	3
Dentistry	2	2	2	2
Increases in Service Stock	**93**	**95**	**103**	**97**
Fabrication of jewelry	82	84	88	85
Increase in bars, coins, medals	11	11	15	12
Sources	**(105)**	**(93)**	**(98)**	**(99)**
Mining	(74)	(73)	(73)	(73)
Recycled scrap	(18)	(19)	(19)	(19)
Official sales	(16)	(3)	(6)	(8)
Gold loan repayments	2	2	1	2
Discrepancy	**1**	**(14)**	**(17)**	**(10)**

Chart C

Net Sales of Official Gold and Large Sales by Selected Holders
(millions of troy ounces)

Year	IMF	United States	Canada	Belgium	Netherlands	Other *	Net Sales*	Cumulative Net Sales
1974	--	--	--	--	--	0.6	0.6	--
1975	--	1.3	--	--	--	(1.0)	0.3	0.9
1976	3.9	--	--	--	--	(2.0)	1.9	2.8
1977	6.0	--	--	--	--	2.6	8.6	11.4
1978	5.9	4.1	--	--	--	1.6	11.6	23.0
1979	5.5	11.8	--	--	--	0.2	17.5	40.5
1974-79	21.3	17.2	--	--	--	2.0	40.5	40.5
1980	2.2	--	1.2	--	--	(10.8)	(7.4)	33.1
1981	--	--	--	--	--	(8.9)	(8.9)	24.2
1982	--	--	--	--	--	(2.7)	(2.7)	21.5
1983	--	--	--	--	--	4.6	4.6	26.1
1984	--	--	--	--	--	2.7	2.7	28.8
1985	--	--	--	--	--	(4.2)	(4.2)	24.6
1986	--	--	--	--	--	(4.7)	(4.7)	19.9
1987	--	--	1.2	--	--	(3.5)	(2.3)	17.6
1988	--	--	1.4	--	--	(10.6)	(9.2)	8.4
1980-88	2.2	--	3.8	--	--	(38.1)	(32.1)	8.4
1989	--	--	1.0	2.0	--	8.8	11.8	20.2
1990	--	--	1.3	--	--	(1.0)	0.3	20.5
1991	--	--	1.8	--	--	(1.0)	0.8	21.3
1992	--	--	3.0	6.5	12.9	(2.4)	20.0	41.3
1993	--	--	3.9	--	--	10.4	14.3	56.6
1994	--	--	--	--	--	2.5	2.5	58.1
1995	--	--	0.5	5.6	--	0.4	6.5	64.6
1996	--	--	--	6.5	9.6	(9.4)	6.7	71.3
1989-96	--	--	11.5	20.6	22.5	8.3	62.9	71.3

*numbers in ( ) are purchases

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CAN GOVERNMENT GOLD BE PUT TO BETTER USE?: QUALITATIVE AND QUANTITATIVE EFFECTS OF ALTERNATIVE POLICIES

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CAN GOVERNMENT GOLD BE PUT TO BETTER USE?:
QUALITATIVE AND QUANTITATIVE EFFECTS OF ALTERNATIVE POLICIES

Dale W. Henderson, John S. Irons, Stephen W. Salant, and Sebastian Thomas*

Abstract: Gold has both private uses (depletion uses and service uses) and government uses. It can be obtained from mines with high extraction costs (about $300 per ounce) or from aboveground stocks with no extraction costs. Governments still store massive stocks of gold. Making government gold available for private uses through some combination of sales and loans raises welfare from private uses by removing two types of inefficiencies. For given private uses, there is a *production inefficiency* if costless government gold is withheld while costly gold is taken from mines. There are *use inefficiencies* if costless government gold is withheld from private users. We assess both qualitatively and quantitatively the gain in welfare and its distribution.

Any policy in a *class* maximizes welfare from private uses. One policy involves selling all government gold immediately. Another involves lending all remaining government gold in every period and selling government gold gradually after some future time. Government uses might require gold *ownership* but not gold *storage*. If so, any loss in welfare from government uses would be much smaller under the policy involving lending and selling gradually.

We construct and calibrate a model of the gold market. We prove that governments always obtain more revenue by making their gold available sooner. For a representative set of parameters, there is a gain in total welfare (discounted economic surplus) of $130 billion (1997 dollars) if governments act now instead of twenty years from now. Before any redistribution, governments gain $128 billion, and the private sector gains $2 billion. According to our measure, a large share of the gain (37%) comes from removing the production inefficiency.

Keywords: gold, exhaustible resource, extraction of a durable

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1 Introduction

Gold has both private uses and government uses. Private uses can be divided into two categories: *depletion uses* which reduce the stock and *service uses* which do not. Depletion uses include amounts used up in the electronics industry, other industries, and dentistry; service uses include services derived from the stocks of jewelry and bars, coins, and medals. Government uses include services derived from stocks held as monetary assets, as "war chests," and as strategic materials.

Gold for any use can be obtained either from mines or from aboveground stocks. The average cost of extracting gold from the mines is approximately \$300 per ounce.[1] However, the average cost of "extracting" gold from aboveground stocks is zero. Governments still store massive aboveground stocks. Making these stocks available for private uses through some combination of sales and loans raises welfare from private uses. We assess both qualitatively and quantitatively the gain in welfare from private uses and its distribution among the participants in the gold market. Of course, making government stocks available for private uses raises total welfare only if the gain in welfare from private uses exceeds any loss in welfare from government uses.

The gain in welfare from private uses is achieved by reducing two types of inefficiencies. For a given path of private uses, there is a *production inefficiency* if costless government gold is withheld while costly gold is taken from mines. Herfindahl [1967] established as much for resources with only depletion uses and the logic of his argument extends to resources which also have service uses. There are *use inefficiencies* if costless government gold is withheld, even temporarily, from private users who could derive benefit from it.

Governments can make gold available for private uses through a *class* of policies involving equivalent combinations of gold sales and gold loans. A gold loan involves receiving gold today and returning the same amount of gold and a loan fee at some future date. For simplicity, we focus most of our attention on the case of a sale of all government gold. A policy that is equivalent to a sale of all government gold in a given period is a commitment in that period to *lend out* at the beginning of every future period all remaining government gold and to *sell* at the end of every period after some date in the future whatever amount is required to satisfy the demands of depletion users at the price that would have prevailed in that period if all government gold had been sold in the given period. If government uses of gold require *ownership* but not *storage,* any loss in welfare from government uses resulting from making government gold available for private uses would be much smaller under the policy involving lending and gradual sales in the future.

We provide two breakdowns of the total gain in welfare from private uses that re-

[1] We assume that the average cost of extraction remains unchanged in real terms over time. It does not include costs imposed on third-parties (externalities) by gold mining. See footnote 8 for further discussion of extraction costs and externalities.

sults from an earlier versus a later sale of government gold: a breakdown by group of market participants and a breakdown by type of inefficiency reduced. The breakdown by group of market participants is straightforward because the total gain in welfare is calculated by summing the changes in economic surpluses for the five groups of market participants included in our analysis: depletion users, service users, service stock owners, mine owners, and governments. The breakdown by type of inefficiency reduced is accomplished by constructing a hypothetical policy under which market prices and the welfares of all private agents are the same as in the case of a later sale, but the present value of government revenue is higher. The increase in the present value of government revenue is the measure of the gain from reducing the production inefficiency.[2] Under the hypothetical policy, governments engineer a postponement of costly extraction. They replace the extra gold that would be taken from belowground during the initial mining phase in the case of a later sale with gold from their aboveground stocks until the period in which these stocks are exhausted, the period in which mining ceases, or the period before the later sale whichever comes first and replenish these stocks by taking gold from belowground in the next period.

One of our comparisons is an unanticipated sale of government gold in period 0 versus a sale in period 20.[3] (We calibrate our model so that a period corresponds to a year.) The estimated gain in total welfare (discounted economic surplus) is $130 billion 1997 dollars with a representative set of parameters. Government revenues are $128 billion higher, *an increase of 60%.* Depletion users and service stock owners gain $8 billion and $51 billion, respectively, but service users and mine owners lose $6 billion and $51 billion, respectively. In the case in which government gold is made available by lending and gradual sales in the future, sales start in period 159 if it is made available in period 0 and in period 194 if it is made available in period 20.

The remainder of this paper is divided into five more sections. In section 2 we lay out the building blocks of a model of the gold market with five groups of market participants that we use to analyze alternative government gold policies. The model is different from the conventional model of the extraction of a durable because reductions in the stock result from controllable usage that generates utility ("depletion") instead of from exogenous decay or random loss ("depreciation").[4] Section 3 is a description of two competitive equilibria: (1) one in which government gold is withheld forever and (2) one in which government gold is sold at some time while mining is in progress. Section 4 contains a proof of the proposition that the sooner governments sell their gold the higher is their revenue. In section 5 we calibrate the model and estimate the effects of alternative government gold policies. Our conclusions are in section 6.

[2]Of course, the increase in government revenue can be distributed to private agents.

[3]We focus on the case of an unanticipated sale for simplicity. See footnote 14, 18, and 21.

[4]The key contributions to the literature on the extraction of a durable include Karp [1993], Levhari and Pindyck [1981], Malueg and Solow [1990], and Stewart [1980]. None allows for depletion that generates utility or, in a market context, is sensitive to price.

2 The Building Blocks of the Model

In this section we present the building blocks of the model of the gold market.

2.1 The Behavior of Private Market Participants

In our model there are four groups of private market participants: depletion users, service users, service stock owners, and mine owners. We consider the behavior of each group in turn. There is a finite horizon of $T+1$ periods running from period 0 to period T.

2.1.1 Depletion Users

In each period of the horizon, depletion users buy q_t units of gold at the price of P_t per unit and derive utility from consuming them. According to the inverse demand function for depletion users,

$$P_t = P(q_t, t), \qquad t = 0, ..., T, \tag{1}$$

the price they are willing to pay falls with the quantity depleted, is non-decreasing with time, and goes to infinity as the quantity depleted goes to zero:[5]

$$\frac{\partial P_t}{\partial q_t} < 0, \qquad \frac{\partial P_t}{\partial t} \geq 0, \qquad P(0,t) = \infty, \qquad t = 0, ..., T. \tag{2}$$

It is sometimes convenient to employ the demand function for depletion users,

$$q_t = q(P_t, t), \qquad t = 0, ..., T. \tag{3}$$

[5]Consider a set of assumptions that yields an inverse demand function which has the properties listed in equation (2). Assume that the representative depletion user's problem is to find the

$$\max_{q_{i,t}\geq 0,\ y^q_{i,t}\geq 0} \quad \sum_{t=0}^{T} \beta^t \left[u(q_{i,t}) + y^q_{i,t}\right], \qquad \text{subject to} \quad \sum_{t=0}^{T} I^t \left[\bar{y}^q_{i,t} - P_t q_{i,t} - y^q_{i,t}\right] \geq 0,$$

where $u' > 0$, $u'' < 0$, and $u'(0) = \infty$; $q_{i,t}$, $y^q_{i,t}$, $\bar{y}^q_{i,t}$, β, and I are the gold depletion, the background-good consumption and endowment, and the subjective and market discount factors of depletion user i in period t, respectively; and P_t is the relative price of gold in terms of the background good. Note that the per-period utility function, $u(q_{i,t}) + y^q_{i,t}$, is quasi-linear. Given this problem, it is well known that the inverse demand function for gold for the representative depletion user is

$$P_t = u'(q_{i,t}), \qquad t = 0, \ldots, T.$$

Assume also that in period i there are $\gamma_{q,t}$ identical depletion users where $1 = \gamma_{q,0} \leq \gamma_{q,t} \leq \gamma_{q,t+1}$, $t = 1, \ldots, T$. Then $q_t = \gamma_{q,t} q_{i,t}$, and the inverse demand function for depletion users as a group is

$$P_t = u'\left(\frac{q_t}{\gamma_{q,t}}\right), \qquad t = 0, \ldots, T,$$

which has the properties listed in equation (2).

2.1.2 Service Users

In each period of the horizon, service users borrow A_t units of gold at a loan fee of R_t per unit and derive utility from using them without consuming them. According to the inverse demand function for service users,

$$R_t = R(A_t, t), \qquad t = 0, ..., T, \tag{4}$$

the loan fee they are willing to pay falls with size of the stock borrowed, is non-decreasing with time, and goes to infinity as the stock borrowed goes to zero:[6]

$$\frac{\partial R_t}{\partial A_t} < 0, \qquad \frac{\partial R_t}{\partial t} \geq 0, \qquad R(0, t) = \infty, \qquad t = 0, ..., T. \tag{5}$$

It is sometimes convenient to employ the demand function for service users,

$$A_t = A(R_t, t), \qquad t = 0, ..., T. \tag{6}$$

2.1.3 Service Stock Owners

In every period of the horizon, service stock owners enter with the stock $A_t \geq 0$, loan it out at the loan fee R_t, sell $A_t - A_{t+1} \gtrless 0$ units, and carry forward the stock $A_{t+1} \geq 0$ into period $t+1$. In the initial period, they have the exogenously given stock $\bar{A}$; that is,

$$A_0 = \bar{A}. \qquad \text{(initial condition for } A_t\text{)} \tag{7}$$

In periods $t = 0, ..., T-1$, if they sell one less unit in period t and carry it forward to loan out and sell in period $t+1$, they reduce their discounted revenue by the price in period t, P_t, and raise it by the discounted sum of the loan fee and the price in

[6]Consider a set of assumptions that yields an inverse demand function with the properties listed in equation (4). Assume that the representative service user's problem is to find the

$$\max_{A_{i,t} \geq 0,\ y^A_{i,t} \geq 0} \quad \sum_{t=0}^{T} \beta^t \left[v(A_{i,t}) + y^A_{i,t} \right], \qquad \text{subject to} \quad \sum_{t=0}^{T} I^t \left[\bar{y}^A_{i,t} - R_t A_{i,t} - y^A_{i,t} \right] \geq 0,$$

where $v' > 0, v'' < 0$, and $v'(0) = \infty$; $A_{i,t}$, $y^A_{i,t}$, $\bar{y}^A_{i,t}$, β, and I are the gold stock borrowed, the background-good consumption and endowment, and the subjective and market discount factors of service user i in period t, respectively; and R_t is the gold loan fee in terms of the background good. Note that the per-period utility function, $v(q_{i,t}) + y^A_{i,t}$, is quasi-linear. Given this problem, it is well known that the inverse demand function for gold for the representative service user is

$$R_t = v'(A_{i,t}), \qquad t = 0, \ldots, T.$$

Assume also that in period t there are $\gamma_{A,t}$ identical service users where $1 = \gamma_{A,0} \leq \gamma_{A,t} \leq \gamma_{A,t+1}, t = 1, \ldots, T$. Then $A_t = \gamma_{A,t} A_{i,t}$, and the inverse demand function for service users as a group is

$$R_t = v'\left(\frac{A_t}{\gamma_{A,t}}\right), \qquad t = 0, \ldots, T,$$

which has the properties listed in equation (4).

period $t+1$, $I(R_{t+1}+P_{t+1})$, where $I = \frac{1}{1+i}$ is market discount factor and i is the real interest rate.[7]

Therefore, they are willing to carry forward nothing ($A_{t+1} = 0$) only if $P_t \geq I(R_{t+1} + P_{t+1})$, an indeterminate positive amount ($0 < A_{t+1} < \infty$) only if $P_t = I(R_{t+1}+P_{t+1})$, or an infinite amount ($A_{t+1} = \infty$) only if $P_t \leq I(R_{t+1}+P_{t+1})$. Since $P_t \gtreqless I(R_{t+1} + P_{t+1})$ implies $P_t - IP_{t+1} \gtreqless IR_{t+1}$, the carry forward conditions for A_t can written as,

$$\begin{array}{rll} A_{t+1} = 0, & \text{only if} & P_t - IP_{t+1} \geq IR_{t+1}, \\ 0 < A_{t+1} < \infty, & \text{only if} & P_t - IP_{t+1} = IR_{t+1}, \quad \text{(carry forward conditions for } A_t\text{)} \\ A_{t+1} = \infty, & \text{only if} & P_t - IP_{t+1} \leq IR_{t+1}, \end{array} \tag{8}$$

for periods $t = 0, ..., T-1$.

Since period T is the last period of the finite horizon, service stock owners can neither loan out nor sell gold carried forward into period $T+1$. Therefore, they do not want to carry forward anything into period $T+1$, that is,

$$A_{T+1} = 0. \qquad \text{(terminal condition for } A_t\text{)} \tag{9}$$

We refer to equation (9) as the terminal condition for A_t.

2.1.4 Mine Owners

Mine owners enter period t with the stock H_t and sell $h_t \in [0, H_t]$ units. At the beginning of the horizon, they have the exogenously given stock $\bar{H}$ in the mines; that is,

[7] The problem faced by a representative service stock owner is to find the

$$\sup_{A_{i,t+1} \geq 0} \sum_{t=0}^{T} I^t \left[R_t A_{i,t} + P_t(A_{i,t} - A_{i,t+1}) \right], \quad \text{subject to } A_{i,0} = \bar{A}_i,$$

where $A_{i,t}$ and $\bar{A}_i$ are the service stock in period t and the exogenous initial stock for owner i, respectively. Since the service stock owner's problem is linear, values that satisfy the first-order conditions must be optimal. The derivatives of the objective function with respect to $A_{i,t+1}$, $t = 0, ..., T$, respectively, are

$$-I^t P_t + I^{t+1}(R_{t+1} + P_{t+1}) = I^t[-P_t + I(R_{t+1} + P_{t+1})], \quad t = 0, ..., T-1, \quad \text{and } - I^T P_T,$$

which are independent of the levels of $A_{i,t+1}$, $t = 0, ..., T$, respectively. Therefore, the first order conditions imply

$$\begin{array}{rll} A_{i,t+1} = 0 & \text{only if} & P_t - IP_{t+1} \geq IR_{t+1}, \\ 0 < A_{i,t+1} < \infty & \text{only if} & P_t - IP_{t+1} = IR_{t+1}, \quad t = 0, ..., T-1, \quad \text{and } A_{T+1} = 0. \\ A_{i,t+1} = \infty & \text{only if} & P_t - IP_{t+1} \leq IR_{t+1}. \end{array}$$

$$H_0 = \bar{H} \,. \qquad \text{(initial condition for } H_t\text{)} \tag{10}$$

They will only sell in periods in which net revenue is non-negative; that is,

$$h_t > 0 \text{ only if } P_t - c \geq 0, \qquad \text{(positive mining condition)} \tag{11}$$

where c is the constant marginal cost of extraction.[8] If net revenue is non-negative in at least one period, they are willing to extract everything in the mines in the period or periods which have the highest discounted net revenue $I^t(P_t - c)$.[9]

If they mine one less unit in period t and one more unit in period $t+1$, they reduce their net revenue by the price in period t minus the constant per unit extraction cost, $P_t - c$, and increase their net revenue by the discounted difference between the price

[8]For simplicity, we assume that in each period all ore deposits have the same constant marginal cost of extraction and that in every period the constant marginal cost of extraction is the same in real terms. Our analysis could be modified to allow for a marginal cost of extraction for each deposit that rises with the amount extracted in each period or for ore deposits with different constant marginal costs of extraction. The marginal cost of extraction will probably increase in real terms in the future as low-cost reserves are used up and only less accessible ore remains.

The cost of extraction does not include costs imposed on third-parties (externalities) arising from mercury, arsenic, and cyanide poisoning associated with gold mining in different regions of the world nor the costs from human wastes of the miners. Descriptions of gold mining in the United States and Brazil by Duncan (1997) and the World Bank (1991), respectively, indicate that these externalities can be very large.

[9]The problem faced by the representative mine owner is to

$$\max_{h_{i,t} \geq 0} \sum_{t=0}^{T} I^t(P_t - c)h_{i,t}, \quad \text{subject to} \quad \bar{H}_i - \sum_{t=0}^{T} h_{i,t} \geq 0.$$

The first-order conditions are

$$h_{i,t} \geq 0, \quad I^t(P_t - c) - \kappa \leq 0, \quad h_{i,t}[\, I^t(P_t - c) - \kappa] = 0, \qquad t = 0, \ldots, T,$$

$$\kappa \geq 0, \quad \bar{H}_i - \sum_{t=0}^{T} h_{i,t} \geq 0, \quad \kappa[\bar{H}_i - \sum_{t=0}^{T} h_{i,t}] = 0,$$

where κ is the multiplier appended to the reserve constraint. Since the mine owner's problem like the service stock owner's problem of the previous footnote is linear, values that satisfy the first-order conditions must be optimal. If $P_t - c < 0$, $t = 0, \ldots, T$, then since $\kappa \geq 0$ from the last first-order condition, it follows from the first $T+1$ first-order conditions that $I^t(P_t - c) - \kappa < 0$ and, therefore, that $h_{i,t} = 0$, $t = 0, \ldots, T$. If $P_t - c \geq 0$ in at least one period, κ must be set equal to the highest value of $I^t(P_t - c)$, mining may occur in any period in which $I^t(P_t - c) - \kappa = 0$, and everything in the mine is sold. If κ is set lower than this value, then at least one of the first $T + 1$ first-order conditions is violated. If κ is set higher than this value, then the first T first-order conditions imply that $I^t(P_t - c) - \kappa < 0$ and that $h_{i,t} = 0$, $t = 0, \ldots T$, but the last first-order condition implies that $\bar{H}_i - \sum_{t=0}^{T} h_{i,t} = 0$, so there is an inconsistency. If κ is set equal to this value, then the first T first-order conditions imply that $I^t(P_t - c) - \kappa = 0$ and that $h_{i,t} \geq 0$ in every period in which $I^t(P_t - c)$ attains its highest value and also that $I^t(P_t - c) - \kappa < 0$ and that $h_{i,t} = 0$ in every other period, and the last first-order condition implies that $\bar{H}_i - \sum_{t=0}^{T} h_{i,t} = 0$.

in period $t+1$ and the extraction cost, $I(P_{t+1}-c)$. They mine in period t but not in period $t+1$ only if $(P_t - c) \geq I(P_{t+1}-c)$, mine in both period t and period $t+1$ only if $(P_t - c) = I(P_{t+1}-c)$, and mine in period $t+1$ but not in period t only if $(P_t - c) \leq I(P_{t+1}-c)$. Since $(P_t - c) \gtreqless I(P_{t+1}-c)$ implies $P_t - IP_{t+1} \gtreqless Iic$, the adjacent period mining conditions can be rewritten as

$$\begin{array}{lll} h_t > 0,\ h_{t+1} = 0 & \text{only if} & P_t - IP_{t+1} \geq Iic, \\ h_t > 0,\ h_{t+1} > 0 & \text{only if} & P_t - IP_{t+1} = Iic, \quad \text{(adjacent period mining, 1)} \\ h_t = 0,\ h_{t+1} > 0 & \text{only if} & P_t - IP_{t+1} \leq Iic. \end{array} \qquad (12)$$

2.2 Mining Activity and the Size of the Service Stock

In this subsection we first derive some relationships between mining activity and the size of the service stock in competitive equilibrium. We then use these relationships and two additional properties of the model to establish three propositions.

We assume that $R(0,t) = \infty$ and that the initial stock of gold, $A_0 = \bar{A}$, is positive but finite. Therefore, in order for the market for gold loans to be in equilibrium, lenders must carry forward a positive but finite service stock into every period after the initial period $(0 < A_t < \infty, t = 1, ..., T)$. According to the middle condition in equations (8), they are willing to satisfy this requirement only if the price in period t exceeds the discounted value of the price in period $t+1$ by exactly the discounted value of the loan payment in period $t+1$ for all pairs of adjacent periods in the finite horizon:

$$P_t - IP_{t+1} = IR_{t+1}, \qquad t = 0, ..., T-1. \qquad \text{(loan recursion)} \qquad (13)$$

We call the condition in equation (13) the loan recursion. Note that the loan recursion implies that when the price rises, it must rise by less than the rate of interest.

In what follows we often refer to the condition that must be fulfilled for there to be mining in adjacent periods, the middle condition in equations (12):

$$P_t - IP_{t+1} = Iic. \qquad \text{(mining recursion)} \qquad (14)$$

We call the condition in equation (14) the mining recursion.

If both the loan recursion and the mining recursion hold, then $A_{t+1} = A^*_{t+1}$where A^*_{t+1} is defined implicitly by

$$R(A^*_{t+1}, t+1) = ic. \quad \text{(definition of } A^*_{t+1}) \qquad (15)$$

Combining the loan recursion with the second inequalities in equations (12) implies that the adjacent period mining conditions can be rewritten as

$$\begin{array}{lll} h_t > 0,\ h_{t+1} = 0 & \text{only if} & A_{t+1} \leq A^*_{t+1}, \\ h_t > 0,\ h_{t+1} > 0 & \text{only if} & A_{t+1} = A^*_{t+1}, \quad \text{(adjacent period mining, 2)} \\ h_t = 0,\ h_{t+1} > 0 & \text{only if} & A_{t+1} \geq A^*_{t+1}, \end{array} \qquad (16)$$

since $R(A_t, t)$ is strictly decreasing in its first argument.

We now call attention to two additional properties of the model. First, A_t is less than A_{t-1} when there is no mining in period $t-1$ because depletion is positive in every period given our assumption that $P(0,t) = \infty$. That is, $A_t = A_{t-1} + h_{t-1} - q(P_{t-1}, t-1) < A_{t-1}$ if $h_{t-1} = 0$ because $q(P_{t-1}, t-1) > 0, t = 1, \ldots, T$. Second, A_t^* increases weakly over time. Our assumptions in equation (5) that $R(A_t, t)$ is decreasing in A_t and is non-decreasing in t imply that as time passes it takes a non-decreasing service stock to keep the loan fee constant at ic as required by the definition of A_t^* in equation (15).

Using the conditions in equations (16) and the two additional properties just stated, we can establish three propositions. The first proposition is that if mining ceases in some period $\hat{t}$, then it cannot resume in any period after that and is stated formally as

Proposition 1: If $h_{\hat{t}-1} > 0$ and $h_{\hat{t}} = 0$, then $h_{\hat{t}+k} = 0$, $k \in 1, \ldots, T - \hat{t}$.

Proof. Assume that mining ceases in period $\hat{t}$ so that $h_{\hat{t}-1} > 0$ and $h_{\hat{t}} = 0$. Then, the conditions in equation (16) imply that $A_{\hat{t}} \leq A_{\hat{t}}^*$. Suppose, contrary to the proposition, that mining resumes in period $\hat{t} + k$ so that $h_{\hat{t}+k-1} = 0$ and $h_{\hat{t}+k} > 0$ for some $k \in 1, \ldots, T - \hat{t}$. Then, the conditions in equation (16) imply that $A_{\hat{t}+k} \geq A_{\hat{t}+k}^*$. But this is impossible because A_t is strictly decreasing when there is no mining in period $t-1$ and A_t^* is weakly increasing. Therefore, $h_{\hat{t}+k} = 0$, $k \in 1, \ldots, T - \hat{t}$.

The first proposition implies that all mining takes place in one unbroken string of periods. We refer to an unbroken string of periods with mining as a mining phase. In our terminology, the first proposition implies that there is only one mining phase.

The second proposition is that if there is no mining in some period $\hat{t}$, but mining begins in some later period, then $A_{\hat{t}} > A_{\hat{t}}^*$, and is stated formally as

Proposition 2: If $h_{\hat{t}} = 0$ but $h_{\hat{t}+k-1} = 0$ and $h_{\hat{t}+k} > 0$ for some $k \in 1, \ldots, T - \hat{t}$, then $A_{\hat{t}} > A_{\hat{t}}^*$.

Proof. Assume that $h_{\hat{t}} = 0$. Assume also that mining starts in period $\hat{t} + k$ so that $h_{\hat{t}+k-1} = 0$ and $h_{\hat{t}+k} > 0$ for some $k \in 1, \ldots, T - \hat{t}$. Then the third line of equation (16) implies that $A_{\hat{t}+k} \geq A_{\hat{t}+k}^*$. Suppose, contrary to the proposition, that $A_{\hat{t}} \leq A_{\hat{t}}^*$. Then, there is a contradiction: $A_{\hat{t}+k} < A_{\hat{t}+k}^*$ because A_t is strictly decreasing when there is no mining in period $t-1$, and A_t^* is weakly increasing. Therefore, it must be that $A_{\hat{t}} > A_{\hat{t}}^*$.

The third proposition is that if there is no mining in the some period $\hat{t}$, and there is no more gold belowground in that period, then $A_{\hat{t}+k} < A_{\hat{t}+k}^*$, $k = 1, \ldots, T - \hat{t}$, and is stated formally as

Proposition 3: If $h_{\hat{t}} = 0$ and $H_{\hat{t}} = 0$, $\hat{t} = 1, \ldots, T$, then $A_{\hat{t}+k} < A^*_{\hat{t}+k}$, $k = 1, \ldots, T - \hat{t}$.

Proof. Assume that mining ends in some period $\hat{t} - m - 1$ so that $h_{\hat{t}-m-1} > 0$ and $h_{\hat{t}-m} = 0$, $m = \hat{t} - 1, \ldots, 0$. Then the third line of (16) implies that $A_{\hat{t}-m} \leq A^*_{\hat{t}-m}$. Since A_t is strictly decreasing when there is no mining in period $t-1$, and A^*_t is weakly increasing, $A_{\hat{t}+k} < A^*_{\hat{t}+k}$ for $k = 1, \ldots, T - \hat{t}$.

3 Competitive Equilibrium and a Sale of Government Gold

In this section we describe competitive equilibrium in the gold market with no sale of government gold and with a sale of government gold during the mining phase.

3.1 No Sale of Government Gold

In this subsection, we describe competitive equilibrium with no sale of government gold, that is, when government gold is withheld forever. Our description is based on two results from the preceding section and two simplifying assumptions. The two results are (1) that the loan recursion must be satisfied in all periods and (2) that there is only one mining phase. The two simplifying assumptions are (1) that there is mining in at least period 0 and period 1 and (2) that T is large enough that mining stops before period T.[10]

[10]The first simplifying assumption has several implications: $P_0 - c = I(P_1 - c) \geq 0$, $A_1 = A^*_1$, $q(P_0, 0) + A^*_1 > \bar{A}$ so that $h_0 = q(P_0, 0) + A^*_1 - \bar{A} > 0$, $\bar{H} > h_0$, and $q(P_1, 1) + A_2 > A^*_1$ so that $h_1 = q(P_1, 1) + A_2 - A^*_1 > 0$.

The second simplifying assumption can be stated in specific terms. Suppose there is mining at $t = 0, 1, ..., T$. Then at t

$$P_t = \frac{ic}{1+i} + \frac{ic}{(1+i)^2} + \ldots + \frac{ic}{(1+i)^{T-t}} + \frac{P(A^*_T + h_T)}{(1+i)^{T-t}} = c + \frac{P(A^*_T + h_T) - c}{(1+i)^{T-t}}.$$

Note that

$$\frac{ic}{1+i} + \frac{ic}{(1+i)^2} + \ldots + \frac{ic}{(1+i)^{T-t}} = c - \frac{ic}{(1+i)^{T-t}}\left(\frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots\right) = c - \frac{c}{(1+i)^{T-t}}.$$

Cumulative demand less the initial stock is, therefore,

$$\sum_{t=0}^{T} q\left(c + \frac{P(A^*_T + h_T) - c}{(1+i)^{T-t}}\right) - \bar{H} - \bar{A} \geq \sum_{t=0}^{T} q\left(c + \frac{P(A^*_T) - c}{(1+i)^{T-t}}\right) - \bar{H} - \bar{A}.$$

The inequality follows since $h_T \geq 0$ and $P'(\cdot) < 0$. If only the stock A^*_T is sold to depletion users at T, the price is weakly higher and demand is weakly lower than if $h_T \geq 0$ is also sold. For T sufficiently great, the right hand side of the last equation would be positive. An increase in T increases the number of terms in the sum of demands and increases the magnitude of each existing term because it weakly lowers $P(A^*_T)$ and strictly raises $(1+i)^{T-t}$ and, therefore, strictly increases $q(\cdot)$. Let $\grave{T}$ be the smallest T such that the right hand side of the last equation is strictly positive. In specific terms, our second simplifying assumptions is that $T \geq \grave{T}$.

The two results and two additional assumptions imply that the economy passes through two phases. These two phases are shown by the solid lines in the five panels of Figure 1.[11] In the first phase, mining is going on. The first phase runs from period 0 to some period $\acute{t}-1$, $\acute{t}\geq 2$ where $\acute{t}=32$ in Figure 1. During the first phase, the path of the price (top left) and the path of the loan fee (middle left) satisfy both the loan recursion and the mining recursion, so mining (bottom right) is going on. Depletion (top right) first rises and then falls because the effect of the shift outward in demand resulting from population growth first outweighs the effect of the price increase and then is outweighed by it. In periods 1 through $\acute{t}-1$, the service stock (middle right) carried into the period t is equal to A_t^*, $(A_t = A_t^*,\ t=1,...,\acute{t})$. In periods 1 through $\acute{t}-2$, service stock owners loan out A_t^* and make purchases equal to any increase in A_t^* $(A_{t+1}^*-A_t^*\geq 0)$. Mine owners sell an amount equal to the sum of depletion and any increase in A_t^*; that is, $h_t = q(P_t,t)+A_{t+1}^*-A_t^*$, $t=1,...,\acute{t}-2$. In period 0, service stock owners loan out A_0, and make purchases, no purchases or sales, or sales depending on whether $A_1^*-A_0 \gtreqless 0$. Mine owners sell an amount equal to the sum of depletion plus $A_1^*-A_0\gtreqless 0$; that is, $h_0=q(P_0,0)+A_1^*-A_0$. In period $\acute{t}-1$, service stock owners loan out $A_{\acute{t}}^*$ and make purchases, no purchases or sales, or sales depending on whether $A_{\acute{t}+1}-A_{\acute{t}}^*\gtreqless 0$, where $A_{\acute{t}+1}^*-A_{\acute{t}}^*\geq A_{\acute{t}+1}-A_{\acute{t}}^*\gtreqless 0$. Mine owners sell an amount equal to the sum of depletion plus $A_{\acute{t}+1}-A_{\acute{t}}^*\gtreqless 0$; that is, $h_{\acute{t}} = q(P_{\acute{t}},\acute{t})+A_{\acute{t}+1}-A_{\acute{t}}^*$. The total sales of mine owners during the first phase exhaust the mines; that is, $\sum_{t=0}^{\acute{t}-1} h_t = \sum_{t=0}^{\acute{t}-1} q(P_t,t)+A_{\acute{t}}-A_0=\bar{H}$. The exhaustion

If the left hand side of the last equation is equal to zero, mining begins in period 0 and continues to the end of the horizon. The paths of prices and loan fees satisfy both the loan recursion and the mining recursion over the whole horizon. In periods 1 through $T-1$, service stock owners loan out A_t^* and make purchases equal to $A_{t+1}^*-A_t^*\gtreqless 0$. Mine owners sell $h_t = q(P_t,t)+A_{t+1}^*-A_t^*$, $t=1,...,T-1$. In period 0, service stock owners loan out A_0 and make purchases equal to $A_1^*-A_0\geq 0$. Mine owners sell $h_0=q(P_0,0)+A_1^*-A_0$. In period T, service stock owners loan out A_T^* and then sell it. Mine owners sell $h_T=q(P_T,T)-A_T^*$. Cumulative depletion is equal to the sum of the initial belowground and service stocks. Therefore, there is no service stock remaining in period $T+1$, that is $A_{T+1}=0$.

We believe that the case in which mining stops before the end of the horizon is the more relevant case. Our belief is based on two considerations. The first consideration is theoretical. Mining must cease in an infinite horizon model. Suppose to the contrary that mining did not cease. Then, in any period t, the loan fee would be equal to ic, and the price would be equal to the sum of discounted loan fees over the infinite horizon which would be equal to c:

$$P_t=\sum_{s=1}^{\infty} I^s R_{t+s} = ic\sum_{s=1}^{\infty} I^s = c.$$

But this is impossible, for if the price were constant at c, the sum of depletion demands would eventually exceed $\bar{H}+\bar{A}$. We use a finite horizon model only for convenience and believe that it is not useful to focus attention on a result that can arise only in a finite horizon model. The second consideration is empirical. As is reported below, in all of the simulations with calibrated versions of our model mining ceases before period 50 of a 400 period horizon.

[11] Figure 1 is constructed using a set of parameters that we refer to as the *reference set* and describe in detail later.

of the mines in predicted to be abrupt only because of the of the approximation of a constant unit cost of extraction.[12]

In the second phase there is no more gold in the mines. The second phase runs from period $\acute{t}$ to period T, the paths of prices and loan fees satisfy the loan recursion but not the mining recursion. In periods $\acute{t}$ through T, service stock owners loan out A_t and make sales equal to depletion; that is, $A_{t+1} - A_t = -q(P_t, t), t = \acute{t}, \ldots, T$. The total sales of service stock owners during periods $\acute{t}$ through T exhaust $A_{\acute{t}}$; that is, $\sum_{t=\acute{t}}^{T} q(P_t, t) = A_{\acute{t}}$. Therefore, there is no service stock remaining in period $T+1$; that is $A_{T+1} = 0$.

3.2 A Sale of Government Gold During the Mining Phase

In this subsection, we describe competitive equilibrium when government gold is made available for private uses during the mining phase. Our description is based on the results and additional assumptions used in the last section as well as on one other additional assumption and a theorem that is proved in Appendix A.[13]

Government gold can be made available for private uses through a *class* of policies involving equivalent combinations of gold sales and gold loans. All of the policies in this class induce the same paths for the service stock and depletion and, therefore, for mining. For simplicity, in this section we focus our attention on a policy at one extreme of this class, a sale of all government gold in a given period. At the other extreme is policy under which government sales are postponed for as long as possible. Under this policy, governments make a commitment in the given period that at the beginning of every future period they will *lend out* all the gold they have left and that in periods after the entire service stock available at the beginning of the given period has been used up and in which the mines are closed or exhausted they will *sell* to depletion users in each period the amount they demand at the price that would have resulted if there had been a sale of all government in the given period.

We describe competitive equilibrium in the gold market when there is an unanticipated sale of all government gold in period $\breve{t}$ during the mining phase.[14] The one other additional assumption that we use is that the stock of government gold, $\bar{G}$, is

[12]See footnote 8.

[13]The approach of this section can be used with minor modification to explain what happens in the more straightforward case in which government gold is sold after the mines are exhausted.

[14]In our qualitative and quantitative analysis, we assume for simplicity that government sales are completely unanticipated (or, equivalently, are anticipated to occur with probability zero). If instead the market assigned *non-negligible* probability to having government gold sold in each period, then the approach outlined in Salant and Henderson [1978] would be more appropriate. As explained in footnotes 18 and 21, assuming that each period the market anticipates that government gold will be sold with a non-negligible probability given that it has not been sold already would not affect our *qualitative* conclusions. However, changing the expectations assumption would reduce the *quantitative* estimate of the gain in total welfare from private uses that would result from making government gold available. The higher the probability of a sale, the larger the reduction.

large enough that no mining occurs in period $\check{t}$.[15] The theorem that we use is referred to as the Main Theorem. According to this theorem, for two situations in the gold market, if the *total* stock is weakly larger in the first situation, the *service* stock is strictly larger in the first situation, and the *service* stock in the first situation is large enough that the mines are closed, then the price is lower in the first situation. Since a sale of government gold during the mining phase raises both the total stock and the service stock and raises the service stock above the level that is consistent with the mines being open, it must reduce the price.

We provide a formal statement of the theorem here and restate and prove it in Appendix A. Let $P_t = P(A_t, H_t, t)$ represent the equilibrium price in period t if the service stock and the belowground stock at the start of the period are A_t and H_t, respectively, and let $\hat{P}_t \ = P(\hat{A}_t, \hat{H}_t, t)$ and $\tilde{P}_t = P(\tilde{A}_t, \tilde{H}_t, t)$ represent the prices corresponding to the triple $\hat{A}_t$, $\hat{H}_t$, and t and the triple $\tilde{A}_t$, $\tilde{H}_t$, t, respectively. Now we can state the Main Theorem:

Main Theorem: If $\hat{A}_t + \hat{H}_t \geq \tilde{A}_t + \tilde{H}_t$, and $\hat{A}_t > \tilde{A}_t \geq A_t^*$ then $\hat{P}_t < \tilde{P}_t$ for any $t = 0, \ldots, T$.

It is useful to confirm that this theorem implies that a sale of government gold in period $\check{t}$ causes the price to fall if $A_{\check{t}} = A_{\check{t}-1} + h_{\check{t}-1} - q(P_{\check{t}-1}, \check{t} - 1) + \bar{G} > A_{\check{t}}^*$ whether or not the mines are exhausted. Suppose that when government gold is sold the service stock and the belowground stock are $\tilde{A}_{\check{t}}$ and $\tilde{H}_{\check{t}}$, respectively. The price in the absence of the sale would be $\tilde{P}_{\check{t}} = P(\tilde{A}_{\check{t}}, \tilde{H}_{\check{t}}, \check{t})$. The price in the presence of a sale is $f_{\check{t}} = \hat{P}_{\check{t}} = P(\hat{A}_{\check{t}}, \hat{H}_{\check{t}}, \check{t})$ where $\hat{A}_{\check{t}} = \tilde{A}_{\check{t}} + \bar{G}$ and $\hat{H}_{\check{t}} = \tilde{H}_{\check{t}}$. As long as $\bar{G} > 0$, the Main Theorem implies that the price must fall.

The additional assumption and the Main Theorem imply that when government gold is sold in period $\check{t}$ during the mining phase, the economy passes through three phases after government gold is sold. These three phases are shown by the dotted lines in the five panels of Figure 1 where $\check{t} = 20$.[16] In the first phase there is no mining. The first phase runs from period $\check{t}$ to some period $\check{t} - 1$, $\check{t} - 1 \geq \check{t}$, where $\check{t} = 41$ in Figure 1. The price (top left) drops from about \$382 to about \$317 per ounce in period $\check{t}$ and remains below the no sale path thereafter. Depletion uses (top right) are

[15] In specific terms, it is that $\bar{G} \geq A_{\acute{t}}^* + q(c, \acute{t}) - A_1^* + \mid \bar{A} - A_1^* \mid$, where period $\acute{t}$ is the last period in which there is mining if government gold is withheld forever. Suppose that all government gold is sold in period $\check{t} \leq \acute{t}$. Under the additional assumption $A_{\check{t}+1} = \bar{G} + A_{\check{t}} + h_{\check{t}} - q(P_{\check{t}}, \check{t}) > \bar{G} + A_{\check{t}} - q(c, \check{t}) > A_{\acute{t}+1}^* + q(c, \acute{t}) - A_1^* + \mid \bar{A} - A_1^* \mid + A_{\check{t}} - q(c, \check{t}) > A_{\acute{t}+1}^* \geq A_{\check{t}+1}^*$, $\check{t} = 0, \ldots, \acute{t}$ since $A_{\check{t}} = \bar{A}$ for $\check{t} = 0$, $A_{\check{t}} = A_{\check{t}}^* \geq A_1^*$ for $\check{t} = 1, \ldots, \acute{t}$, and $q(c, t)$ is weakly increasing in t. Now, suppose that $h_{\check{t}} > 0$. Then, the top two lines in (16) imply that $A_{\check{t}+1} \leq A_{\check{t}+1}^*$. But this is impossible. Therefore, $h_{\check{t}} = 0$.

[16] We assume that T is sufficiently large that the mines are exhausted before the end of the horizon. We believe that this case is the more relevant case for reasons given in footnote 10.

higher in the period of the sale and thereafter because the price is lower. In the first phase, the paths of prices and loan fees (middle left) satisfy the loan recursion but not the mining recursion, so the mines are closed (bottom right). The existence of a first phase is guaranteed by our assumption that $\bar{G} \geq A^*_{\tilde{t}} + q(c, \tilde{t}) - A^*_1 + \mid \bar{A} - A^*_1 \mid$. The mines close in period $\breve{t}$, and the service stock (middle right) increases sharply in period $\breve{t}+1$. During this phase while the mines are closed $(h_t = 0, t = \breve{t}, \ldots, \check{t}-1)$ the service stock falls between periods because of depletion.[17] However, the service stock remains above A^*_t which is represented by the rising part of the solid line, the rising part of the dotted line, and the rising dashed line segment connecting these parts in Figure 1; that is $(A_t > A^*_t, t = \breve{t}+1, \ldots, \check{t}-1)$.

In the second phase, there is mining. The second phase runs from period $\check{t}$ to some period $\vec{t}-1$ when the mines are exhausted where $\vec{t} = 60$ in Figure 1. The second phase when government gold is sold in period $\breve{t}$ is qualitatively identical to the first phase when government gold is withheld forever. That is, period $\check{t}$ after a sale in period $\breve{t}$, is qualitatively identical to period 0 when government gold is withheld forever; periods $\check{t}+1$ through $\vec{t}-2$ after government gold is sold in period $\breve{t}$ are qualitatively identical to periods 1 through $\acute{t}-2$ when government gold is withheld forever, and period $\vec{t}-1$ after government gold is sold in period $\breve{t}$ is qualitatively identical to period $\acute{t}-1$ when government gold is withheld forever.

In the third phase, there is no more gold in the mines. The third phase when government gold is sold in period $\breve{t}$, which runs from period $\vec{t}$ to period T, is qualitatively identical to the second phase when government gold is withheld forever, which runs from period $\acute{t}$ to period T.

As noted earlier, the postponement of costly mining is one source of the increase in welfare from private uses that results from a sale of government gold. With no sale, mining (bottom right) continues to occur after period $\breve{t}$ and falls slowly until period $\acute{t}-1$ when the mines are exhausted. By contrast, with a sale in period $\breve{t}$, the mines shut down in that period, reopen again in period $\check{t}$ and are exhausted in period $\vec{t}-1$. During the first $(A_{t+1} \geq A^*_{t+1})$, second $(A_{t+1} = A^*_{t+1})$, and third $(A_{t+1} \leq A^*_{t+1})$ phases, the net revenue from mining a unit in period t is less than, equal to, and greater than the discounted net revenue from mining a unit in period $t+1$. Therefore, the representative mine owner finds it at least as attractive to mine later in every period of the first phase, finds it equally attractive to mine in all the periods of the second phase, and finds it as least as attractive to mine earlier in every period of the third phase. In the first phase net revenue from mining a unit in period t is less than or equal to the discounted net revenue from mining a unit in period $t+1$ because tomorrow's price must be high relative to today's in order to induce service stock owners to hold gold. This inducement is necessary since the service stock is high and, therefore, the loan fee is low. The reopening and exhaustion of the mines

[17]This statement must be modified if $\bar{G}$ is so small that mining stops for only one period.

are predicted to be abrupt only because of the approximation of a constant unit cost of extraction.

4 How Timing Affects Revenue from a Government Sale

In this section we first illustrate and then prove the proposition that the sooner governments sell their gold the higher their revenue.[18] First we illustrate the proposition. Using Figure 2, we explain why government revenue is higher with a sale in period 0 than with a sale in period 20.[19] With a sale in period 0 (dotted line) the price falls from \$350 to \$309, increases at a rate less than the rate of interest, and reaches \$332 in period 20. It increases at a rate less than the rate of interest because the return to holding gold includes not only price appreciation but also the loan fee If governments invest their revenue (dot/dash line) it grows at the real rate of interest and reaches \$506 per ounce in period 20, a level considerably above \$332. If governments do not sell until period 20 (solid line) the price is higher until then; as a consequence, depletion is smaller. Therefore, after a sale in period 20, the total stock is larger than it would be if there were a sale in period 0, so the price, \$317, is lower than \$332. It follows that with a sale in period 0, government revenue is \$189 per ounce higher in period 20, as indicated by the gap between the dot-dash line and the solid line.

Now we prove the proposition. Let $f_{\hat{t},\hat{t}+j}$ denote the price that results in period $\hat{t}+j$ if all government gold is sold in period $\hat{t}$. When their gold is sold in period $\hat{t}$, governments earn $f_{\hat{t},\hat{t}}\bar{G}$. [20] When their gold is sold in period $\hat{t}+1$, governments earn $\frac{f_{\hat{t}+1,\hat{t}+1}}{(1+r)}\bar{G}$. We prove that

$$f_{\hat{t},\hat{t}} > \frac{f_{\hat{t},\hat{t}+1}}{(1+r)} > \frac{f_{\hat{t}+1,\hat{t}+1}}{(1+r)}.$$

The first inequality follows since $f_{\hat{t},\hat{t}+1} = f_{\hat{t},\hat{t}}(1+r) - R(A_{\hat{t}+1}, \hat{t}+1)$ and $R(\cdot,\cdot) > 0$.

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[18] Postponing the time at which government gold is sold also reduces the present value of government revenue if sales are anticipated with non-negligible probability by the market. The proof is very similar to what is outlined above. For a proof in a model which abstracts from both mining costs and the service-flow from gold, see Salant and Henderson [1978], p.636.

[19] Figure 2, like Figure 1, is constructed using a set of parameters that we refer to as the *reference set* and describe in detail later.

[20] It is obvious that governments earn $f_{\hat{t},\hat{t}}\bar{G}$ if they sell all their gold in period $\hat{t}$. What is a little less obvious is that they also earn $f_{\hat{t},\hat{t}}\bar{G}$ from any other policy in the class that results in the same time path for the service stock, depletion, and, therefore, mining as a sale of all gold in period $\hat{t}$. Under any other policy in this class, if a unit of gold is not sold in period $\hat{t}$, it is sold in some later period and loaned out in that and all intervening periods. In a competitive equilibrium the price of a unit of gold in period $\hat{t}$ must be equal to the discounted value of the price of a unit of gold in period $\hat{t}+k$ plus the discounted value of loaning out that unit in periods $t+j, j = 1, \ldots, k$, that is,

$$f_{\hat{t},\hat{t}} = f_{\hat{t},\hat{t}+k} + \sum_{1}^{k} R(A_{\hat{t}+j}, \hat{t}+j).$$

To verify the second inequality we consider two cases. In each, we assume that the magnitude of the government gold sale is so large that no mining will occur in the period in which government gold is sold.

Case 1

In case 1, government gold is sold during the mining phase. In this case, sales at $\hat{t}$ or one period later have the following effects on the service and belowground stocks at $\hat{t}$ +1:

	Service Stock in $\hat{t}+1$	Belowground Stock in $\hat{t}+1$
Government Gold Available in $\hat{t}$	$A^*_{\hat{t}} + \bar{G} - q(f_{\hat{t}}, \hat{t})$	$H_{\hat{t}}$
Government Gold Available in $\hat{t}+1$	$A^*_{\hat{t}} + \Delta - q(P_{\hat{t}}, \hat{t}) + \bar{G}$	$H_{\hat{t}} - \Delta - q(P_{\hat{t}}, \hat{t}+1)$

That is, if government gold sold in period $\hat{t}$, mining ceases, so depletion demand in period $\hat{t}$ must be satisfied from the service stock. If government gold is sold in period $\hat{t}$ +1, demand for the service stock must grow by $\Delta = A^*_{\hat{t}+1} - A^*_{\hat{t}}$, and this growth in stock demand as well as depletion demand must be satisfied from underground stocks. Since the *service* stock in period $\hat{t}$ +1 is larger if government gold is sold in period $\hat{t}$ +1 and so is the *total* stock, the Main Theorem implies that

$$P[A^*_{\hat{t}} + \bar{G} - q(f_{\hat{t}}, \hat{t}), H_{\hat{t}}, \hat{t}+1] > P[A^*_{\hat{t}} + \Delta - q(P_{\hat{t}}, \hat{t}) + \bar{G}, H_{\hat{t}} - \Delta - q(P_{\hat{t}}, \hat{t}), \hat{t}+1].$$

The left-hand side is $f_{\hat{t},\hat{t}+1}$ and the right-hand side is $f_{\hat{t}+1,\hat{t}+1}$. Hence, during the mining phase, $f_{\hat{t},\hat{t}+1} > f_{\hat{t}+1,\hat{t}+1}$ and selling government gold one period sooner increases government revenue.[21]

Case 2

In case 2, government gold is sold outside the mining phase (either because $H_{\hat{t}} = 0$ or $H_{\hat{t}} > 0$ but $A_{\hat{t}+1} > A^*_{\hat{t}+1}$). In this case, selling government gold available in period $\hat{t}$ or one period later will have the following effects on stocks in period $\hat{t}+1$:

[21] If we replace the assumption that private agents expect that government gold will be withheld forever with our 1978 assumption that private agents expect that government gold will be sold in each period with probability α given that it has not been sold before then our conclusion that the revenue generated by a sale of government gold is greater if the gold is actually sold sooner follows from the Main Theorem *provided* a sale would cause the price to fall.

	Service Stock in $\hat{t}+1$	Belowground Stock in $\hat{t}+1$
Government Gold Available in t	$A^*_{\hat{t}} + \bar{G} - q(f_{\hat{t}}, \hat{t})$	$H_{\hat{t}}$
Government Gold Available in $t+1$	$A^*_{\hat{t}} - q(P_{\hat{t}}, \hat{t}) + \bar{G}$	$H_{\hat{t}}$

That is, if government gold is sold in period $\hat{t}$, the price falls and stimulates depletion. If government gold is sold in period $\hat{t}+1$, the price is higher in period $\hat{t}$ and less depletion occurs then. Since the *service* stock in period $\hat{t}+1$ is larger if government gold is sold in period $\hat{t}+1$ and so is the *total* stock, the Main Theorem implies that

$$P[A^*_{\hat{t}} + \bar{G} - q(f_{\hat{t}}, \hat{t}), H_{\hat{t}}, \hat{t}+1] > P[A^*_{\hat{t}} - q(P_{\hat{t}}, \hat{t}) + \bar{G}, H_{\hat{t}}, \hat{t}+1].$$

But the left-hand side is $f_{\hat{t},\hat{t}+1}$, and the right-hand side is $f_{\hat{t}+1,\hat{t}+1}$. Hence, outside the mining phase just as during it, $f_{\hat{t},\hat{t}+1} > f_{\hat{t}+1,\hat{t}+1}$, and selling government gold one period sooner increases government revenue.

5 Quantitative Analysis of Alternative Policies

A principal objective of this paper is to make estimates of the effects of alternative government policies. In order to do so we must calibrate the model. In this section we describe the calibration of the model and discuss our estimates of the effects of alternative policies.

5.1 Calibration of the Model

In this subsection we describe the calibration of the model including our choices for specific functional forms and parameter values. We choose simple functional forms. Above we assume that the real interest rate and the constant marginal cost of mining are stationary over time. Here we assume that the demands of depletion users and service users have the following functional forms:

$$\begin{aligned} q_t &= a\gamma_{q,t}P_t^{-\varepsilon}, \\ A_t &= b\gamma_{A,t}R_t^{-\rho}. \end{aligned} \tag{17}$$

where $\gamma_{q.t}$ and $\gamma_{A,t}$ are given by

$$\begin{aligned} \gamma_{q,t} &= 2 - \gamma_q^t, \quad 0 < \gamma_q < 1, \\ \gamma_{A,t} &= 2 - \gamma_A^t, \quad 0 < \gamma_A < 1. \end{aligned} \tag{18}$$

Given these functional forms, our model has eleven parameters: T, i, c, $\bar{A}$, $\bar{H}$, ε, a, ρ, b, γ_q, and γ_A. If we choose admissible values of the these eleven parameters, it is possible to solve for all the endogenous variables. The values of the parameters that we consider and some data that we use are displayed in Table 1. We set $T = 399$ for all our simulations; that is, we use a horizon of 400 periods, period 0 through period 399, throughout. This horizon is long enough that lengthening it has a negligible impact on solutions for the first hundred periods.

For the other ten parameters, we choose one parameter set that we refer to as the reference set and then construct additional parameter sets by varying combinations of parameters. Our choices for the values of i, c, $\bar{A}$, γ_q, and γ_A are the same for all parameter sets and are reported in Panel A of Table 1. Our choice for the reference value of $\bar{H}$ is in the line labeled "reference set" in Panel B of Table 1. Our choice for i is well within the range of estimates reported in the macroeconomics literature.

Our choices of values for c, $\bar{A}$, γ_q, γ_A, and the reference value for $\bar{H}$ are based on both data from the gold market and assumptions about population growth. The choices for c is derived from and the choices for $\bar{A}$ and the reference value for $\bar{H}$ are equal to estimates reported in the literature.[22] The literature contains several estimates of c and $\bar{A}$ among which there is reasonably close agreement but relatively few estimates of $\bar{H}$ among which there is considerable divergence.

The values of γ_q and γ_A are chosen such that model predictions are roughly consistent with data for the paths of gold depletion and the service stock. Gold depletion has remained roughly constant or even increased during periods of rising gold prices. In addition, the service stock has grown while mining activity was taking place. In our model, depletion demand can remain roughly constant or rise during periods of rising gold prices only if $\gamma_{q,t}$ increases over time. Also, as explained above, in our model as long as there is mining activity the loan fee must remain constant, so the service stock can increase only if $\gamma_{A,t}$ increases over time. We assume that $\gamma_{q,t}$ and $\gamma_{A,t}$ increase over time and that they are always equal. We interpret them as population indexes, and choose the functional forms for them shown in equations (18) These functional forms imply that they begin at a value of 1 and are close to their asymptotic value of 2 after 50 periods.[23]

We choose the reference values for the remaining four parameters ε, a, ρ, and b so that $\varepsilon = \rho$ and so that the predictions of the model satisfy some conditions. These conditions are that P_0 and q_0, the model predictions for price and depletion in the initial period, be equal to $\bar{P}$ and $\bar{q}$, the "current values" for price and depletion, and that the model prediction for A_0^*, the service stock defined by $R(A_0^*, 0) = ic$, be equal to $\bar{A}$, the current service stock, and, therefore, be consistent with the mining phase being in progress in period 0. Values for $\bar{P}$ and $\bar{q}$ are reported in Panel C of Table

[22]The value for c is a weighted average of cash costs in the different producing countries for 1992 as reported in Gold Fields Mineral Services Ltd. (1993). The value for $\bar{A}$ is the sum of from as of the end of 1995 where values for the individual items are taken from Gold Fields Mineral Services (1996). The reference value for $\bar{H}$ is taken from Bureau of the Mines (1997).

[23]Estimating the path of the world population many years in the future is a very difficult task. Several methods of estimation are considered in Cohen (1995). Our assumption that world population levels off at twice its current value by about 2050 is consistent with one of these methods.

Table 1: **Values of Parameters and Data**

Panel A: Parameters That Are the Same in All Simulations

$T = 399^a$	$i = .025$
$c = 300^b$	$\bar{A} = 2468^c$
$\gamma_q = 0.96$	$\gamma_A = 0.96$

Panel B: Parameters That Change Among Simulations

	Parameters		
parameter set	$\bar{H}$	ε	ρ
reference set	2292^c	0.98	0.98
$\rho <$ *reference* ρ	2292^c	2.22	0.70
$\rho >$ *reference* ρ	2292^c	0.52	1.20
$\bar{H} <$ *reference* $\bar{H}$	1719^c	1.51	1.51
$\bar{H} <$ *reference* $\bar{H}$	2865^c	0.63	0.63

Panel C: Data on Initial Gold Price and Depletion Uses

$\bar{P} = 350^b$	$\bar{q} = 10.3^c$ per period

Panel D: Data on Government Gold Stocks

$\bar{G} = 1107^c$	U.S. Gold $= 262^c$

[a] periods [b] dollars per troy ounce [c] millions of troy ounces

1[24]. First, we pick a trial value of $\varepsilon = \rho$. Next, we choose a so that $P_0 = \bar{P}$ and $q_0 = \bar{q}$ by solving for a in $\bar{q} = a\bar{P}^{-\varepsilon}$ and choose b so that $A_0^* = \bar{A}$ by solving for b in $\bar{A} = b(ic)^{-\rho}$. Then, we compute the paths of all the variables in every period up to and including period T as well as A_{T+1}. If $A_{T+1} \approx 0$, the trial value of $\varepsilon = \rho$ is consistent with the remaining reference values, so it is taken as the reference value; otherwise, we continue to pick new trial values until we find one for which $A_{T+1} \approx 0$. In order to make sure that the $\varepsilon = \rho$ pair is unique, we try several widely spaced starting values for $\varepsilon = \rho$. The reference values for $\varepsilon = \rho$ are shown in the line labeled "reference set" in Panel B of Table 1.

Many assumptions are made in constructing the reference set of parameters. In order to explore the sensitivity of our quantitative results to changes in two of these assumptions, we construct four additional sets of parameters by varying combinations of parameters. The values of $\bar{H}$, ε, and ρ for these sets of parameters are shown in Panel B of Table 1. We begin by changing the assumption that $\varepsilon = \rho$ since there seems to be no obvious reason why the two elasticities should be the same. We construct two sets of parameter values in which $\varepsilon \neq \rho$, one with ρ lower than its reference value and one with ρ higher. We assume that a change in ρ is reflected in changes in ε, a, and b. First, we choose a value for ρ. Next, we choose a trial value of ε. Then, we choose a so that $P_0 = \bar{P}$ and $q_0 = \bar{q}$ by solving for a in $\bar{q} = a\bar{P}^{-\varepsilon}$ and choose b so that $A_0^* = \bar{A}$ by solving for b in $\bar{A} = b(ic)^{-\rho}$. If $A_{T+1} \approx 0$, the trial value of ε is consistent with the remaining values in the parameter set; otherwise, we continue to pick new trial values of ε until we find one for which $A_{T+1} \approx 0$. We proceed by changing the assumption about $\bar{H}$ since even the most careful estimates of $\bar{H}$ are very problematic. We construct two more sets of parameter values by reverting to the assumption that $\varepsilon = \rho$ and considering two alternative values of $\bar{H}$, one that is 25 percent lower than its reference value and one that is 25 percent higher. We choose values of $\varepsilon = \rho$, a, and b for which $P_0 = \bar{P}$, $q_0 = \bar{q}$, $A_0^* = \bar{A}$, and $A_{T+1} \approx 0$ given the alternative hypothetical values of $\bar{H}$.

5.2 Estimates of the Effects of Alternative Policies

In this subsection we describe simulations of the effects of alternative government gold policies using five sets of parameters, the reference set and the four additional sets described in the last subsection and displayed in Table 1. We consider what happens if there is an unanticipated sale of all government gold at different points in time under all the parameter sets and if there is an unanticipated immediate sale of only U.S. gold under the reference set. The values of $\bar{G}$ and U.S. gold are given in

[24]The value of $\bar{P}$ is a round number close to an average of market prices for the fist half of 1997. The value of $\bar{q}$ is an average of depletion uses for the years 1993, 1995, and 1996 as reported in Gold Fields Mineral Services Ltd. (1996).

Panel D of Table 1.[25]

5.2.1 The Effects of Selling Government Gold under the Reference Set

We begin by considering the effects of selling government gold at time $\hat{t}$, $\hat{t} = 0, \ldots, T$ under the reference set of parameters. Of course, since the sale of government gold is unanticipated, in all periods before $\hat{t}$ the paths of all the variables are the same as they would be if government gold were withheld forever. To determine the values of the variables from period $\hat{t}$ on we choose a trial value $P_{\hat{t}}$. The amount of government gold sold is always large enough that it is not optimal for any gold to be extracted from the mines in the period of the sale or soon thereafter for reasonable values of the other parameters. Therefore, $A_{\hat{t}+1}$, the service stock in the period following the sale is obtained by adding the government stock, $\bar{G}$, to the difference between the service stock and depletion in period $\hat{t}$, that is $A_{\hat{t}+1} = A_{\hat{t}} - q(P_{\hat{t}}) + \bar{G}$. To determine whether the trial value of $P_{\hat{t}}$ is consistent with the values of $A_{\hat{t}}$, $H_{\hat{t}}$, and the values of i, c, γ_q, γ_A ε, a, ρ, and b in the reference set, we compute the paths of all the variables in every period up to and including period T as well as A_{T+1}. If $A_{T+1} \approx 0$, the trial value of $P_{\hat{t}}$ is the new equilibrium price in period $\hat{t}$; otherwise, we continue to pick new trial values until we find one for which $A_{T+1} \approx 0$. In order to make sure that the new equilibrium price is unique, we try several widely spaced starting values for $P_{\hat{t}}$.

The five panels of Figure 3 show the effects on the gold market of two extreme government gold policies, no sale of any government gold (the solid lines) and an immediate sale of all government gold (the dotted lines), under the reference set of parameters. With an immediate sale the price (top left panel) drops at once from \$350 to about \$309 per ounce and remains below the no sale path thereafter; as a result, depletion uses (top right panel) are higher in every period. With an immediate sale the loan fee (middle left panel) is lower initially and in most periods and is never higher: the service stock (middle right panel) is higher initially and in most periods and is never lower. With no sale, mining (bottom right panel) continues to occur and falls slowly until period 31 when the mines are projected to be exhausted. By contrast, with an immediate sale, the mines shut down at once, reopen again in period 11 and are exhausted in period 58.

5.2.2 Two Breakdowns of the Welfare Gain

We provide estimates of two breakdowns of the gain in total welfare from selling government gold earlier rather than later: a breakdown by group of market participants and a breakdown by type of inefficiency reduced. The first breakdown is by group of market participants. The gain in total welfare is actually obtained by adding up the gains in welfare of the five groups of market participants. For depletion users

[25] The values for $\bar{G}$ and U.S. gold are the holdings of all governments and international institutions and the holdings of the U.S government as of the end of November 1996 as reported in International Monetary Fund (1997).

and service users, the gains in any period are the increases in the areas under their demand curves above the price and rental in that period, respectively, and the total gains are the discounted sums of these increases in areas. For the government, service stock owners, and mine owners, the gains in any period are the increases in revenues in that period, and the total gains are the discounted sums of the these increases in revenues.

The second breakdown is by type of inefficiency reduced. We separate the gain in total welfare from an earlier versus a later sale of government gold into the gain from reducing the production inefficiency and the gain from reducing use inefficiencies. In order to do so, we construct a hypothetical policy under which market prices and the welfares of all private agents are the same as with a later sale, but the present value of government revenue is higher. Under the hypothetical policy, governments sell to mine owners the additional amount of gold they would have mined in the initial mining phase with a later sale between the times of the earlier sale and the later sale until government stocks are exhausted and mine owners sell this gold to private users. In exchange, governments receive (1) an amount per ounce equal to the cost of extraction and (2) title to an equal amount of underground gold. Governments invest their proceeds at the prevailing rate of interest and extract all the underground gold they have acquired in the period after their stocks run out.[26] There is an increase in the present value of government revenue because the present value of receipts exceeds the present value of the cost of extracting the underground gold. The increase in the present value of government revenue is a measure of the part of the gain from an earlier sale that results from reducing the production inefficiency.[27] The remainder of the gain from an earlier sale is the part that results from reducing use inefficiencies.

The upper left and right panels of Figure 4 show the estimated effects on welfare from private uses for three comparisons of alternative government selling policies under the reference set of parameters. The first columns in the upper left and upper right panels show how welfare changes with an immediate sale of all government gold versus no sale. Total welfare increases by \$368 billion. Although total welfare increases, the breakdown by groups shows that not all groups of market participants are better off. Most of the increase in welfare (93%) takes the form of government revenue in the first instance. Depletion users and service users gain, but service stock owners and mine owners lose. The breakdown by type of inefficiency reduced shows that 13% comes from eliminating the production inefficiency. Under the hypothetical

[26]If the time of the later sale comes before government stocks are exhausted, governments extract all the belowground gold which they have acquired in the period of the later sale.

[27]Our measure of the gain from reducing the production inefficiency is a bound in the sense that we assume that governments mine all the belowground gold to which they have obtained title in the period right after their stocks are exhausted. If governments mined the underground gold more slowly, for example, at the rate at which they acquired it, the measure of the gain from reducing the production inefficiency would be larger.

policy, governments use their gold to replace what would have been taken from the mines in periods 0 - 10 and part of what would have been taken from the mines in period 11 before running out and extract the belowground gold they have acquired in period 12 to replenish their stocks.

The second columns in the upper left and right panels show how welfare changes with a sale of all gold in period 20 versus no sale. The pattern of gains and losses is similar, but the magnitudes are somewhat different. By period 20 mine owners have extracted most of the gold in the mines (73%), so the gain from reducing the production inefficiency is significantly less and is a smaller share (7%) of the total. Depletion users and service users get government gold somewhat later with a sale in period 20, so the welfare gain from reducing use inefficiencies is also significantly less. Under the hypothetical policy, governments use part of their gold to replace what would have been taken from the mines in periods 20 - 31 and extract the belowground gold they have acquired in period 32 to replenish their stocks.

Some may find it implausible that governments would never sell their gold, so in the third column of the upper left and right panels we present the welfare effects of an immediate sale of all government gold versus a sale in period 20. Total welfare is $130 billion higher with an immediate sale because inefficiencies are eliminated at once. Since the gain from reducing the cost inefficiency is the same as for the comparison of an immediate sale versus no sale but the size of the total welfare increase is much less, the share of the gain that results from reducing the cost inefficiency (37%) is much greater.

Figure 5 shows the estimated effects on welfare from private uses of the sale of all government gold at every time $\hat{t}$, $\hat{t} = 0, \ldots, T$ versus no sale under the reference set of parameters. The differences in total welfare, government revenue, and depletion users' welfare fall monotonically with delay, and the difference in mine owners' welfare rises monotonically with delay. In contrast, if the sale occurs when the mines are open the difference in service users' welfare and the difference in service stock owners' welfare rises and falls monotonically with delay, respectively, and if the sale occurs thereafter, the difference in service users' welfare and the difference in service stock owners' welfare falls and rises monotonically with delay, respectively. The difference in service users' welfare and stock owners' welfare do not change monotonically with the date of the government sale because of our assumption that the demand schedules shift outward over time. Given these assumed nonstationarities postponing a government sale has both positive and negative effects of on the welfare of the two types of agents as long as the mines are open. We will provide a more detailed explanation of these effects in the next version of this paper.

5.2.3 Sensitivity Analysis

We investigate the sensitivity of our quantitative results to the choices of values for some parameters by making some estimates using the four alternative sets of parameter values described above. The paths of five variables with no sale (solid lines) and with an immediate sale (dotted lines) for the four alternative sets of parameters are reported in Figures 6 - 9.

The welfare results for the three policy comparisons—an immediate sale versus no sale, a sale in period 20 versus no sale, and an immediate sale versus a sale in period 20—for the four alternative parameter sets are reported in the bottom four panels of Figure 4. For each alternative parameter set for a given policy comparison, the welfare results are reported in terms of (percentage) deviations from the welfare results for the reference set: each entry for an alternative parameter set for a given policy comparison is the deviation of the welfare difference under the alternative parameter set for the given policy comparison from the welfare difference under the reference set for that comparison as a percentage of the *total* welfare difference for that comparison under the reference set. For example, the entry under the $\rho < reference$ ρ parameter set (middle left) for government revenue for the sale in period 20 versus no sale comparison is the deviation of the difference in government revenue for the $\rho < reference\ \rho$ parameter set for a sale in period 20 versus no sale (not shown but equal to \$211 billion) from the difference in government revenue for the reference set for a sale in 20 years versus no sale (\$214 billion from the top left panel of Figure 4) divided by the difference in total welfare under the reference parameter set for a sale in period 20 versus no sale (\$238 billion from the top left panel in Figure 4) and multiplied by 100.[28]

First, consider the alternative parameter set in which $\rho < reference\ \rho$. In this case b and ε must be larger and a must be smaller than their respective reference values in order to satisfy the initial conditions. With no sale the paths of the price, loan fee, and service stock must be the same in every period as with the reference set and depletion must be lower in every period after the first than with the reference set as long as the mines are open for both sets of parameters. The mines close slightly earlier with the reference set. After the mines close for the reference set, the service stock is always a little higher with the reference set, but the loan fee is always a little higher with the alternative set because of the decrease in ρ and increase in a. The price path rises faster for the reference set, and depletion for the alternative set is even farther below depletion for the reference set. Of course, eventually depletion for the alternative set must rise above depletion for the reference set because the same amount of gold must be depleted with both sets.

With an immediate sale the price drop is larger for the alternative set. Depletion demand is lower at all prices above the initial price with the alternative set of parameters. Price is above the initial price for much of the 400 period horizon for both sets of parameters. The same amount of gold must be depleted on both paths. Therefore,

[28]The result of the calculation described in the text is not exactly equal to the entry in Figure 4 because of rounding.

it is not implausible that the price drop at the time of the sale should be larger with the alternative set parameters.

Among the results for the changes in welfare effects, those for the immediate sale versus no sale and for a sale in period 20 versus no sale are easier to interpret. The *total welfare* differences for these two policy comparisons are not very sensitive to the change in ρ; that is, the deviations for the total welfare difference for these comparisons is quite small. The deviations for the differences in government revenue and mine owners' welfare are also quite small. The deviations for the welfare differences for depletions users, service stock owners, and service users are somewhat larger. Since ε is larger, the inverse depletion demand curve is flatter, so the increases in depletion are greater in the earlier periods that count most. Even though the price drops are greater in the first few periods, the price drops are greater in later periods are smaller, so it is not implausible that depletion users gain less. For example, for the immediate sale versus no sale comparison, the price drop is greater for periods 0 through 4, but smaller in period 5 and thereafter. Since ρ is smaller, the inverse service demand curve is steeper, so the loan fee falls more in the early periods that count most. Therefore, it is not implausible that service users gain more, and service stock owners' losses must be greater.

The results for the changes in welfare effects for the immediate sale versus a sale in period 20 are more difficult to interpret. All welfare differences for a sale in a given period versus no sale change monotonically until beyond period 20 for both parameter sets.[29] However, the deviations for the differences for government revenue and the welfare of depletion users for all the comparisons have the same sign, while those for the welfares of service users, service stock owners, and mine owners do not.

The results for the alternative parameter set with $\rho > reference\ \rho$ are different from those for the reference set in ways that are qualitatively opposite from the results for the alternative parameter set with $\rho < reference\ \rho$. However, it is clear that the model is nonlinear. For example, the amount by which the price drop following an immediate sale is more for the parameter set with $\rho < reference\ \rho$ is greater in absolute value than the amount by which the price drop is less for the parameter set with $\rho < reference\ \rho$.

Now, consider the alternative set in which $\bar{H} < reference\ \bar{H}$. In this parameter set $\bar{H}$ is 25% below its reference value, $\rho = \varepsilon$ must be larger, and a and b must be smaller than their respective reference values in order to satisfy the initial and terminal conditions. With an immediate sale the price and loan fee fall less under the alternative set. Depletion demand is lower at every price above $\bar{P}$, and price is above $\bar{P}$ for most of the horizon, but there is less gold to be depleted, so it is not implausible that the price falls less. Service demand is higher at every loan fee below

[29]The welfare differences for the reference set are in Figure 4. The welfare differences for the alternative set with $\rho < reference\ \rho$ are not shown but are available from the authors on request.

ic, so it is not implausible that the loan fee falls less.

Among the results for the changes in welfare effects, those for the immediate sale versus no sale and for a sale in period 20 versus no sale are easier to interpret. The total welfare differences for these two policy comparisons are not very sensitive to the change in $\bar{H}$. Since the price falls by less, the increases in government revenue are larger, and it is not implausible that mine owners' losses are smaller. The inverse depletion demand schedule is flatter, but the price falls by less, so it is not implausible that the increases in the depletion users' welfare are smaller. The inverse service demand schedule is flatter, so the loan fee falls by less in the early periods that count most. Therefore, it is not implausible that the increases in service users' welfare are smaller, and service stock owners' losses must be smaller.

The results for the changes in welfare effects for the immediate sale versus a sale in period 20 are more difficult to interpret. All welfare differences for a sale in a given period versus no sale change monotonically until beyond period 20 for both parameter sets.[30] However, the deviations for the differences in government revenue and in the welfares of depletion users and mine owners for all the comparisons have the same sign, while those for the welfares of service users and service stock owners do not.

The results for the alternative parameter set with $\bar{H} > reference\ \bar{H}$ are different from those for the reference set in ways that are qualitatively opposite from the results for the alternative parameter set with $\bar{H} < reference\ \bar{H}$.

5.2.4 The Incentive for a Government to be the First to Sell

It is fairly obvious that if a country's only objective is to maximize the revenue from an unanticipated sale of its gold, it has an incentive to sell its gold before other governments sell or announce a sale. What is not obvious is the size of the incentive. We consider the case of the United States. Under the reference set, the incentive for the U.S. government to be the first to sell is sizable. The path of the gold price with an immediate sale of only U.S. gold is shown in Figure 10. The gold price drops from $350 to $340 when only U.S. gold is sold instead of to $309 when all government gold is sold. The U.S. government gold stock is 262 million troy ounces or 24% of all government gold. Therefore, U.S. government revenue is about $89 billion when only U.S. gold is sold but only about $81 when all gold is sold, a difference of about $8 billion or about 10%.

6 Conclusions

We have analyzed alternative government gold polices both qualitatively and quantitatively. Welfare from private uses is maximized by making all the gold currently held by governments immediately available to private agents who value its depletion

[30] The welfare differences for the reference set are in Figure 4. The welfare differences for the alternative set with $\bar{H} < reference\ \bar{H}$ are not shown but are available from the authors on request.

uses or service uses. As the simulation results show, the aggregate welfare gains from making government gold available now rather than twenty years from now are quite substantial.

Our analysis shows that the welfare gains from making government gold available sooner are not evenly distributed among the various groups involved. Specifically, we prove analytically that government revenue must increase and in the simulations most of the welfare gains take the form of an increase in government revenue in the first instance. In the simulations depletion users always gain and mine owners always lose from making gold available sooner. However, service users and service stock owners may gain or lose depending on parameters.

In our analysis, government ownership of gold does not contribute directly to the welfare of private agents. There is a view that the welfare of private agents increases with government gold ownership for at least three reasons: (1) gold reserves would be necessary if gold ever again played an important role in international monetary arrangements; (2) gold is an important part of a "war chest" for times of international crisis; and (3) gold is irreplaceable in certain strategic uses. There is almost certainly some truth in this view. However, the importance of gold as possible future reserve asset, as part of a war chest, and as a strategic material has clearly diminished in recent years and will, in all likelihood, continue to diminish.

Of course, any benefits of government *ownership* of gold are lost at once under a policy that involves selling all government gold immediately. However, any such benefits are lost much later under a policy that involves loaning out all government gold immediately and selling is gradually after some date in the future. It is clear that if governments lent out all their gold but wanted to keep open the possibility of using it in a crisis, they would have to structure their loan contracts so that they could get their gold back immediately in a crisis. It is not clear how difficult it would be to incorporate the necessary provisions into loan contracts. In addition, there would also be some costs of administering gold loans and gradual gold sales. However, the difficulties and costs associated with loaning out government gold may well be small enough that it would still be worthwhile for governments to make most or all of their gold available for private uses immediately through gold loans given our estimates of sizable increases in welfare from private uses from making government gold available for private uses.

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7 Appendix A

In this appendix, we prove the Main Theorem of the text, two lemmas, and a Supporting Theorem. First, we prove the Main Theorem.

Main Theorem:

If $\hat{A}_t + \hat{H}_t \geq \tilde{A}_t + \tilde{H}_t$ and $\hat{A}_t > \tilde{A}_t \geq A_t^*$, then $\hat{P}_t < \tilde{P}_t$.

Proof:

Suppose to the contrary that $\hat{P}_t \geq \tilde{P}_t$.

Let $\hat{t}_E$ and $\tilde{t}_E$ denote the ends of the mining phases for the price paths beginning at $\hat{P}_t$ and $\tilde{P}_t$, respectively. That is, for example, $\hat{h}_{\hat{t}_E} > 0$ but $\hat{h}_{\hat{t}_E+1} = 0$. Consequently, $\hat{A}_{\hat{t}_E} = A^*_{\hat{t}_E}$ but $\hat{A}_{\hat{t}_E+1} \leq A^*_{\hat{t}_E+1}$.

According to the Supporting Theorem proved below, $\hat{t}_E < \tilde{t}_E$ and $\hat{P}_s > \tilde{P}_s$ for $s = t+1, \ldots, \hat{t}_E$.

One chain of reasoning leads to the conclusion that at $\hat{t}_E$, the total stock for the price path beginning with $\hat{P}_t$ must be strictly larger than the total stock for the price path beginning with $\tilde{P}_t$. The initial total stock is weakly larger for the price path beginning with $\hat{P}_t$, and there is strictly less depletion for this path.

However, another chain of reasoning leads to the contradictory conclusion that at $\hat{t}_E$ the total stock for the price path beginning with $\hat{P}_t$ must be weakly smaller than the total stock for the other price path. Since $\hat{t}_E < \tilde{t}_E$, then $\hat{H}_{\hat{t}_E+1} = 0 < \tilde{H}_{\hat{t}_E+1}$ and $\hat{A}_{\hat{t}_E+1} \leq \tilde{A}_{\hat{t}_E+1} = A^*_{\hat{t}_E+1}$.

Given these conclusions, the premise that $\hat{P}_t \geq \tilde{P}_t$ cannot be true. Hence, $\hat{P}_t < \tilde{P}_t$.

Next, we establish two lemmas used in the proof of the Supporting Theorem.

Lemma 1:

If $\hat{A}_s \geq \tilde{A}_s$ (with strict inequality for at least one period) for $s = t+1, \ldots, \check{t}$ where $\check{t} \in t+1, \ldots, T$ inclusive of end points and $\hat{P}_t \geq \tilde{P}_t$, then $\hat{P}_{\check{t}} > \tilde{P}_{\check{t}}$.

Proof:

From the loan recursion, equation (13) of the text,

$$P_t = \sum_{s=t+1}^{\check{t}} \frac{R(A_s, s)}{(1+i)^{s-t}} + \frac{P_{\check{t}}}{(1+i)^{\check{t}-t}}. \tag{A.1}$$

But since $R(\hat{A}_s, s) \leq R(\tilde{A}_s, s)$ for $s = t+1, \ldots, \check{t}$ (with strict inequality in at least one period) while $\hat{P}_t \geq \tilde{P}_t$, then $\hat{P}_{\check{t}} > \tilde{P}_{\check{t}}$.

Lemma 2:

If $\hat{H}_t = \tilde{H}_t = 0$ and $\hat{A}_t > \tilde{A}_t$, then $\hat{P}_t < \tilde{P}_t$.

Proof:

Assume to the contrary that $\hat{P}_t \geq \tilde{P}_t$. In the absence of mining $(H_t = 0)$, $A_{t+1} = A_t - q(P_t, t)$. Hence if $\hat{A}_t > \tilde{A}_t$ and $\hat{P}_t \geq \tilde{P}_t$, then $\hat{A}_{t+1} > \tilde{A}_{t+1}$.

Recall that if the stock A_{t+1} is willingly held between periods t and $t+1$, then

$$P_{t+1} = P_t(1+i) - R(A_{t+1}, t+1).$$

Since $\hat{A}_{t+1} > \tilde{A}_{t+1}$ and $R(A_t, t)$ is strictly decreasing in its first argument, it follows that

$$\hat{P}_{t+1} > \tilde{P}_{t+1}.$$

Note that the hypothesis that $\hat{P}_t \geq \tilde{P}_t$ (combined with $\hat{A}_t > \tilde{A}_t$) leads to a similar pair of inequalities one period later. Our arguments can therefore be repeated and will show that as a consequence of our premise,

$$\hat{A}_T > \tilde{A}_T \text{ and } \hat{P}_T > \tilde{P}_T.$$

It follows that the terminal condition that $q(P_T) = A_T$ can not be fulfilled for both price paths. Given that $\hat{A}_T > \tilde{A}_T$, the terminal condition can be fulfilled on both price paths only if $\hat{P}_T < \tilde{P}_T$. Hence, our premise that $\hat{P}_t \geq \tilde{P}_t$ must be false, and in any solution to equilibrium conditions if $\hat{H}_t = \tilde{H}_t = 0$ and $\hat{A}_t > \tilde{A}_t$, then $\hat{P}_t < \tilde{P}_t$.

Finally, we prove the Supporting Theorem.

Supporting Theorem:

If $\hat{A}_t + \hat{H}_t \geq \tilde{A}_t + \tilde{H}_t$, $\hat{A}_t > \tilde{A}_t \geq A_t^*$, and $\hat{P}_t \geq \tilde{P}_t$, then $\hat{t}_E < \tilde{t}_E$ and $\hat{P}_s > \tilde{P}_s$ for $s = t+1, \ldots, \hat{t}_E$.

Proof:

As argued in the proof of Lemma 2, if $\hat{P}_t \geq \tilde{P}_t$, then $\hat{A}_s > \tilde{A}_s$ and $\hat{P}_s > \tilde{P}_s$ for all periods s beginning with period $t+1$ for which there is no mining in period $s-1$ including the period in which mining begins on the price path for which it begins first. It follows that either mining begins later for the price path beginning with $\hat{P}_t$ than for the price path beginning with $\tilde{P}_t$ or mining begins in the same period for both price paths. That is, if we let $\hat{t}_B$ and $\tilde{t}_B$ denote the beginnings of the mining phases for the price paths beginning with $\hat{P}_t$ and $\tilde{P}_t$, respectively, then $\hat{t}_B \geq \tilde{t}_B$.

Whenever mining occurs in both period $s-1$ and period s for a price path, the service stock carried from period $s-1$ to period s is A_s^* and $P_s = IP_{s-1} - R(A_s^*, s)$.

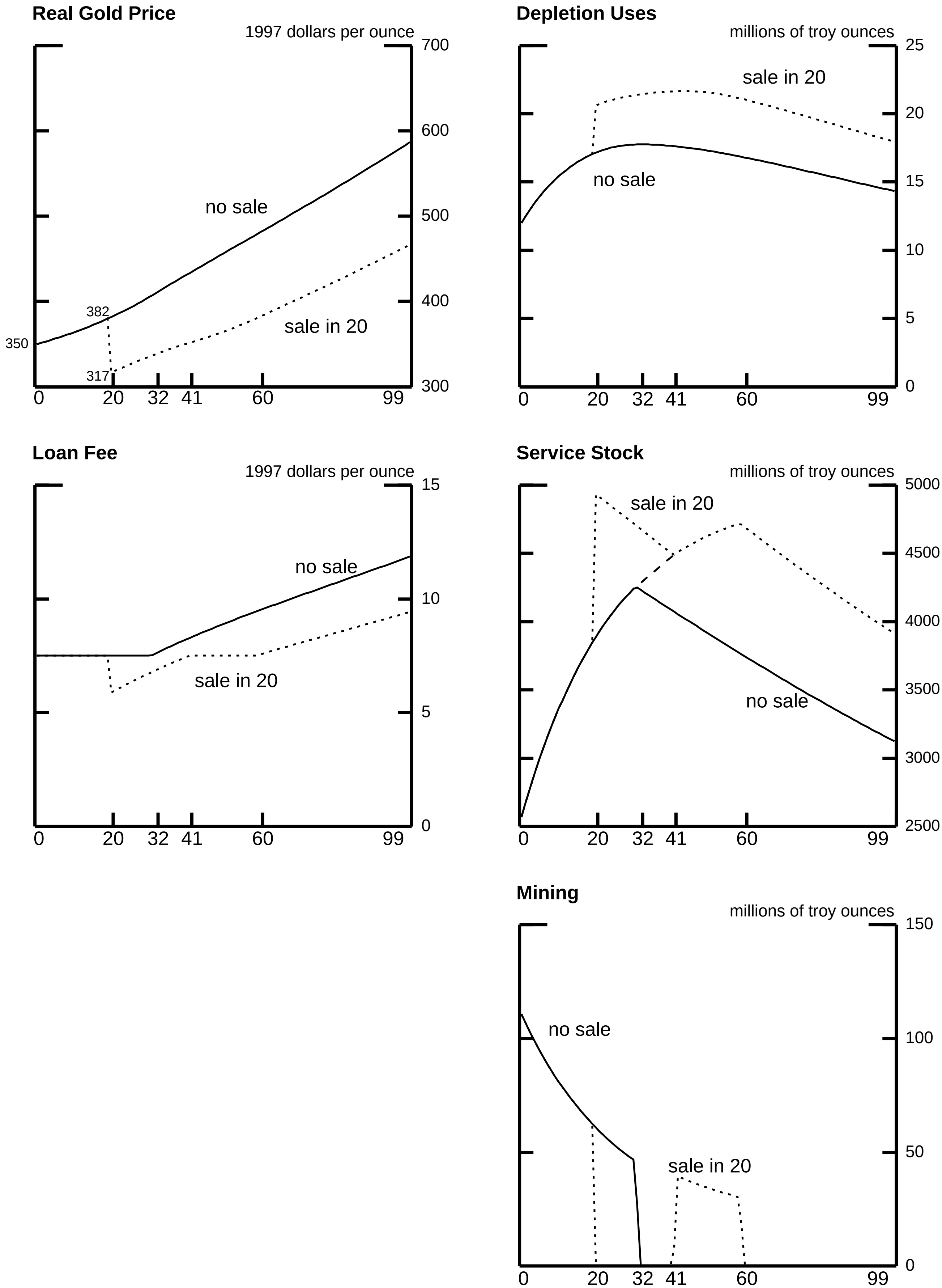
If $\hat{t}_B + 1 = \tilde{t}_B + 1$, $\hat{P}_{t_B} > \tilde{P}_{t_B}$, so if mining continues in period $\hat{t}_B + 1 = \tilde{t}_B + 1$ on both price paths, then $\hat{A}_{\tilde{t}_B+1} = \tilde{A}_{\tilde{t}_B+1} = A^*_{\tilde{t}_B+1}$ so $\hat{P}_{t_B+1} > \tilde{P}_{t_B+1}$. This argument can be repeated for periods $\hat{t}_B + s = \tilde{t}_B + s, s > 1$ in succession; $\hat{P}_{t_B+s-1} > \tilde{P}_{t_B+s-1}$, so if mining continues in period $\hat{t}_B + s = \tilde{t}_B + s$ on both price paths, $\hat{A}_{\tilde{t}_B+s} = \tilde{A}_{\tilde{t}_B+s} = A^*_{\tilde{t}_B+s}$ so $\hat{P}_{t_B+s} > \tilde{P}_{t_B+s}$.

If $\hat{t}_B + 1 > \tilde{t}_B + 1$, $\hat{P}_{t_B} > \tilde{P}_{t_B}$, so if mining continues on the path that begins with $\tilde{P}_t$, then $\hat{A}_{\tilde{t}_B+1} \geq \tilde{A}_{\tilde{t}_B+1} = A^*_{\tilde{t}_B+1}$ depending on whether or not mining begins on the price path that begins with $\hat{P}_t$, so $\hat{P}_{t_B+1} > \tilde{P}_{t_B+1}$. This argument can be repeated for periods $\tilde{t}_B + s, s > 1$ and $\tilde{t}_B + s \leq \hat{t}_B$ in succession; $\hat{P}_{\tilde{t}_B+s-1} > \tilde{P}_{\tilde{t}_B+s-1}$, so if mining continues on the price path that begins with $\tilde{P}_t$, then $\hat{A}_{\tilde{t}_B+s} \geq \tilde{A}_{\tilde{t}_B+s} = A^*_{\tilde{t}_B+s}$ depending on whether or not mining begins on the prices path that begins with $\hat{P}_t$, so $\hat{P}_{t_B+s} > \tilde{P}_{t_B+s}$. In period $\hat{t}_B + 1$, $\hat{P}_{\hat{t}_B} > \tilde{P}_{\hat{t}_B}$, so if mining continues in period $\hat{t}_B + 1$ on both price paths, then $\hat{A}_{\hat{t}_B+1} = \tilde{A}_{\hat{t}_B+1} = A^*_{\hat{t}_B+1}$ so $\hat{P}_{\hat{t}_B+1} > \tilde{P}_{\hat{t}_B+1}$. This argument can be repeated for periods $\hat{t}_B + s, s > 1$ in succession; $\hat{P}_{\hat{t}_B+s-1} > \tilde{P}_{\hat{t}_B+s-1}$, so if mining continues in period $\hat{t}_B + s$ on both price paths, $\hat{A}_{\hat{t}_B+s} = \tilde{A}_{\hat{t}_B+s} = A^*_{\hat{t}_B+s}$, so $\hat{P}_{\hat{t}_B+s} > \tilde{P}_{\hat{t}_B+s}$.

To show that $\hat{t}_E < \tilde{t}_E$, assume the contrary. If $\hat{t}_E \geq \tilde{t}_E$, then $A_s^* = \hat{A}_s \geq \tilde{A}_s$, s= $\tilde{t}_E, \ldots, \hat{t}_E$. After extraction at $\hat{t}_E$, $\hat{H}_{\hat{t}_E+1} = \tilde{H}_{\hat{t}_E+1} = 0$. Since for the path beginning with $\hat{P}_t$ the total initial stock is weakly larger and total depletion is strictly lower, it follows that $\hat{A}_{\hat{t}_E+1} > \tilde{A}_{\hat{t}_E+1}$. Since $\hat{A}_s \geq \tilde{A}_s$ (with strict inequality for at least one period) for $s = t+1, \ldots, \hat{t}_E + 1$ inclusive of end points and $\hat{P}_t \geq \tilde{P}_t$, Lemma 1 implies that $\hat{P}_{\hat{t}_E+1} > \tilde{P}_{\hat{t}_E+1}$. But if $\hat{P}_{\hat{t}_E+1} > \tilde{P}_{\hat{t}_E+1}$ when $\hat{A}_{\hat{t}_E+1} > \tilde{A}_{\hat{t}_E+1}$ and $\hat{H}_{\hat{t}_E+1} = \tilde{H}_{\hat{t}_E+1} = 0$, then Lemma 2 is violated. Therefore, we have established that $\hat{t}_E < \tilde{t}_E$.

We have shown that $\hat{P}_s > \tilde{P}_s$ in any period s beginning with period $t+1$ for which there is no mining in period $s-1$ on the paths beginning with $\hat{P}_t$ and $\tilde{P}_t$ or for which there is mining in both period $s-1$ and period s on the path beginning with $\tilde{P}_t$ as long as if there is mining in period $s-1$ on the path beginning with $\hat{P}_t$ there is also mining in period s. Therefore, since $\hat{t}_E < \tilde{t}_E$, it follows that $\hat{P}_s > \tilde{P}_s$ for $s = t+1, \ldots, \hat{t}_E$.

Figure 1

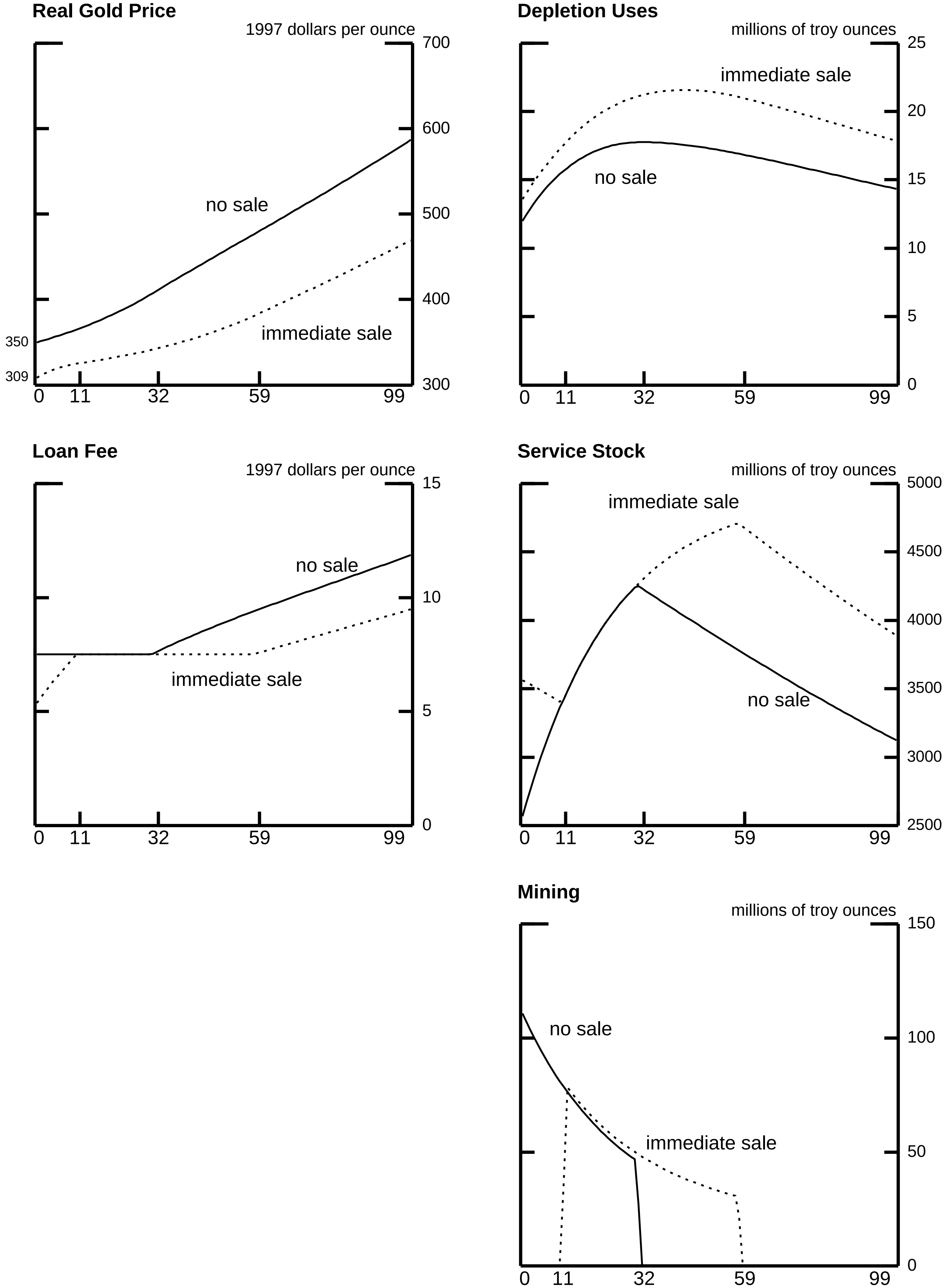


Real Gold Price
1997 dollars per ounce
700
600
500
400
300
no sale
sale in 20
382
350
317
0
20
32
41
60
99
Depletion Uses
millions of troy ounces
25
20
15
10
5
0
sale in 20
no sale
Loan Fee
1997 dollars per ounce
15
10
5
0
no sale
sale in 20
Service Stock
millions of troy ounces
5000
4500
4000
3500
3000
2500
sale in 20
no sale
Mining
millions of troy ounces
150
100
50
0
no sale
sale in 20

Figure 2

Government Revenue with Immediate Sale vs. Sale in 20

Figure 3



Real Gold Price
1997 dollars per ounce
700
600
500
400
300
350
309
no sale
immediate sale
0
11
32
59
99
Depletion Uses
millions of troy ounces
25
20
15
10
5
0
immediate sale
no sale
0
11
32
59
99
Loan Fee
1997 dollars per ounce
15
10
5
0
no sale
immediate sale
0
11
32
59
99
Service Stock
millions of troy ounces
5000
4500
4000
3500
3000
2500
immediate sale
no sale
0
11
32
59
99
Mining
millions of troy ounces
150
100
50
0
no sale
immediate sale
0
11
32
59
99

Figure 4

Estimated Effects on Welfare from Private Uses

	Immediate vs. No	20 vs. No	Immediate vs. 20	Immediate vs. No	20 vs. No	Immediate vs. 20
	Reference Case (billions of 1997 dollars)			Reference Case (% of total)		
Total	**368**	**238**	**130**	**100**	**100**	**100**
Government Revenue	342	214	128	93.0	89.7	99.0
Depletion Users	49	41	8	13.4	17.3	6.4
Service Users	149	155	-6	40.6	-65.2	-5.0
Stock Owners	-102	-153	51	-27.8	-64.2	39.3
Mine Owners	-70	-19	-51	-19.1	-8.0	-39.6
Production Inefficiency	48	16	48	13	7	37
Use Inefficiencies	320	222	82	87	93	63
	$\rho <$ Reference ρ (% change from baseline)			$\rho >$ Reference ρ (% change from baseline)		
Total	**-0.4**	**-0.8**	**0.3**	**0.2**	**0.3**	**-0.1**
Government Revenue	-0.7	-1.1	-0.1	0.3	0.5	0.1
Depletion Users	-3.2	-4.5	-0.7	3.4	5.0	0.4
Service Users	4.8	9.0	-3.0	-4.1	-7.1	1.4
Stock Owners	-1.9	-4.2	2.3	1.0	2.0	-0.9
Mine Owners	0.6	-0.1	1.8	-0.4	0.0	-1.0
Production Inefficiency	0.0	0.6	-0.1	0.0	0.1	0.0
Use Inefficiencies	-0.4	-1.5	0.3	0.2	0.2	-0.1
	$\bar{H} <$ Reference $\bar{H}$ (% change from baseline)			$\bar{H} >$ Reference $\bar{H}$ (% change from baseline)		
Total	**1.1**	**1.5**	**0.3**	**-1.1**	**-1.6**	**-0.4**
Government Revenue	1.7	2.1	0.8	-2.0	-2.6	-0.8
Depletion Users	-3.4	-5.0	-0.4	4.6	6.9	0.3
Service Users	-7.1	-11.2	0.5	7.2	12.0	-1.6
Stock Owners	4.2	8.2	-3.2	-4.9	-9.7	4.0
Mine Owners	5.7	7.4	2.6	-6.1	-8.1	-2.2
Production Inefficiency	0.0	-3.1	0.0	0.0	13.3	0.0
Use Inefficiencies	1.1	4.5	0.3	-1.1	-14.9	-0.4

Figure 5

Effects on Economic Welfare from Private Uses of Sale of All Government Gold

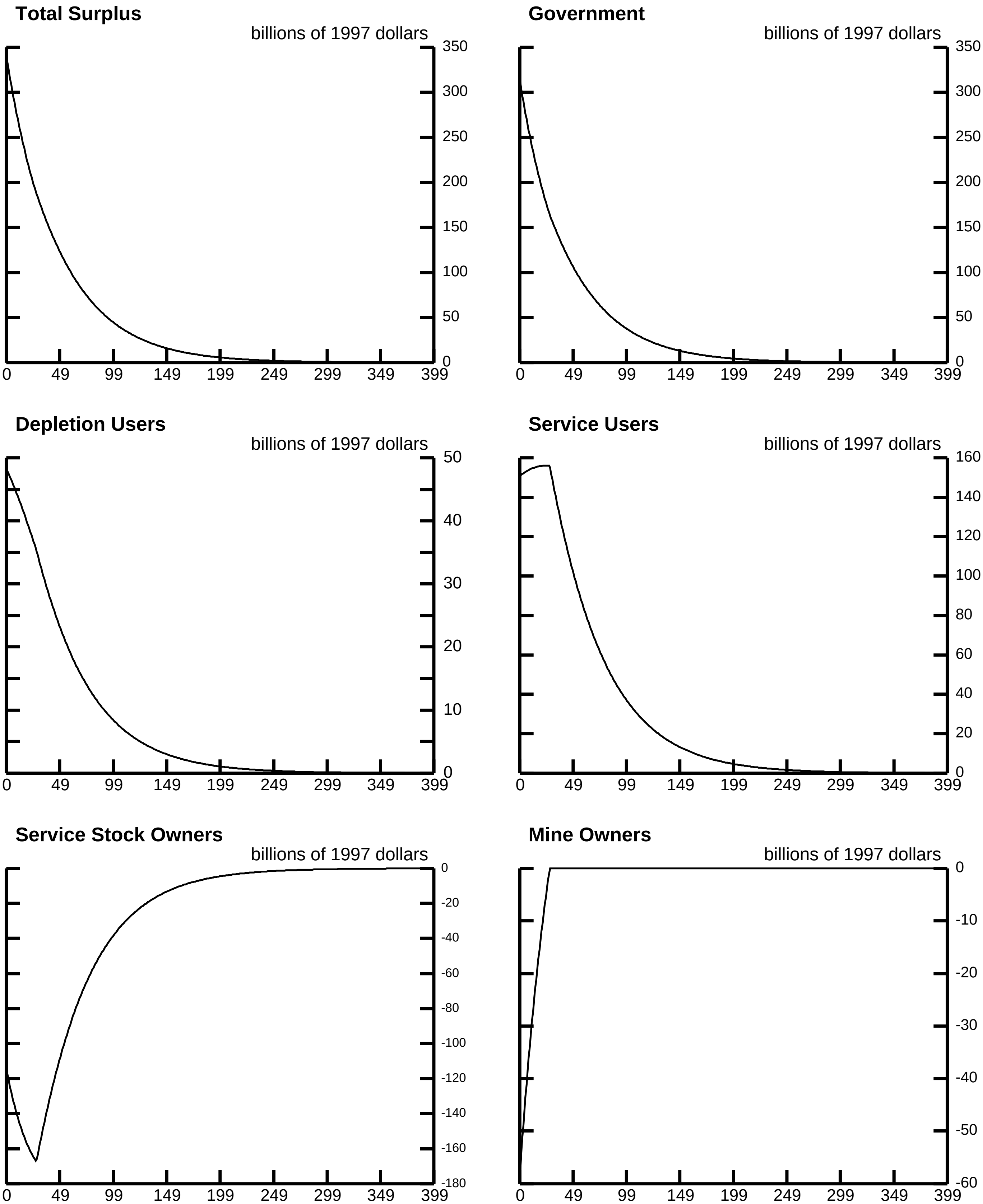


Figure 6

ρ < Reference ρ

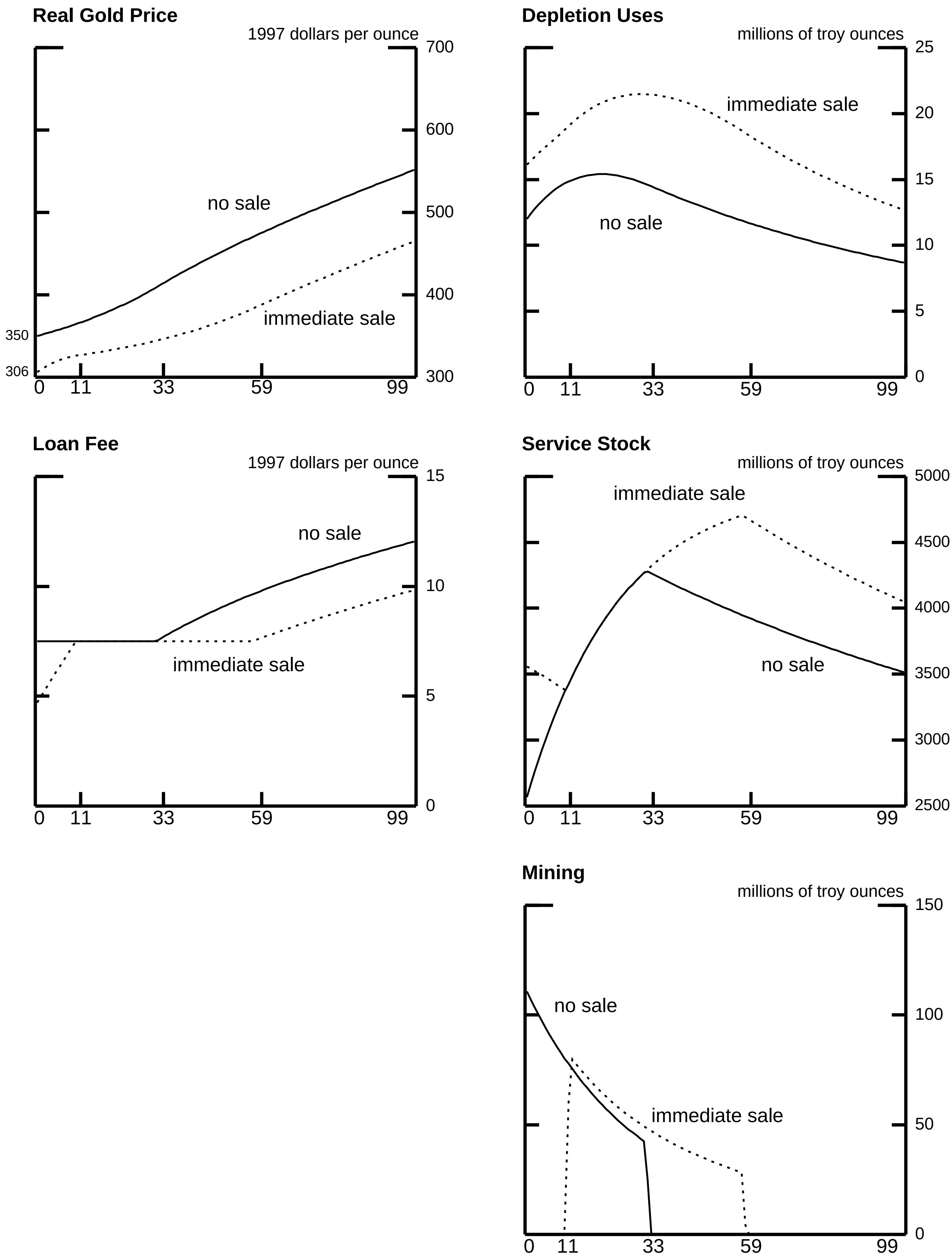


Figure 7

ρ > Reference ρ

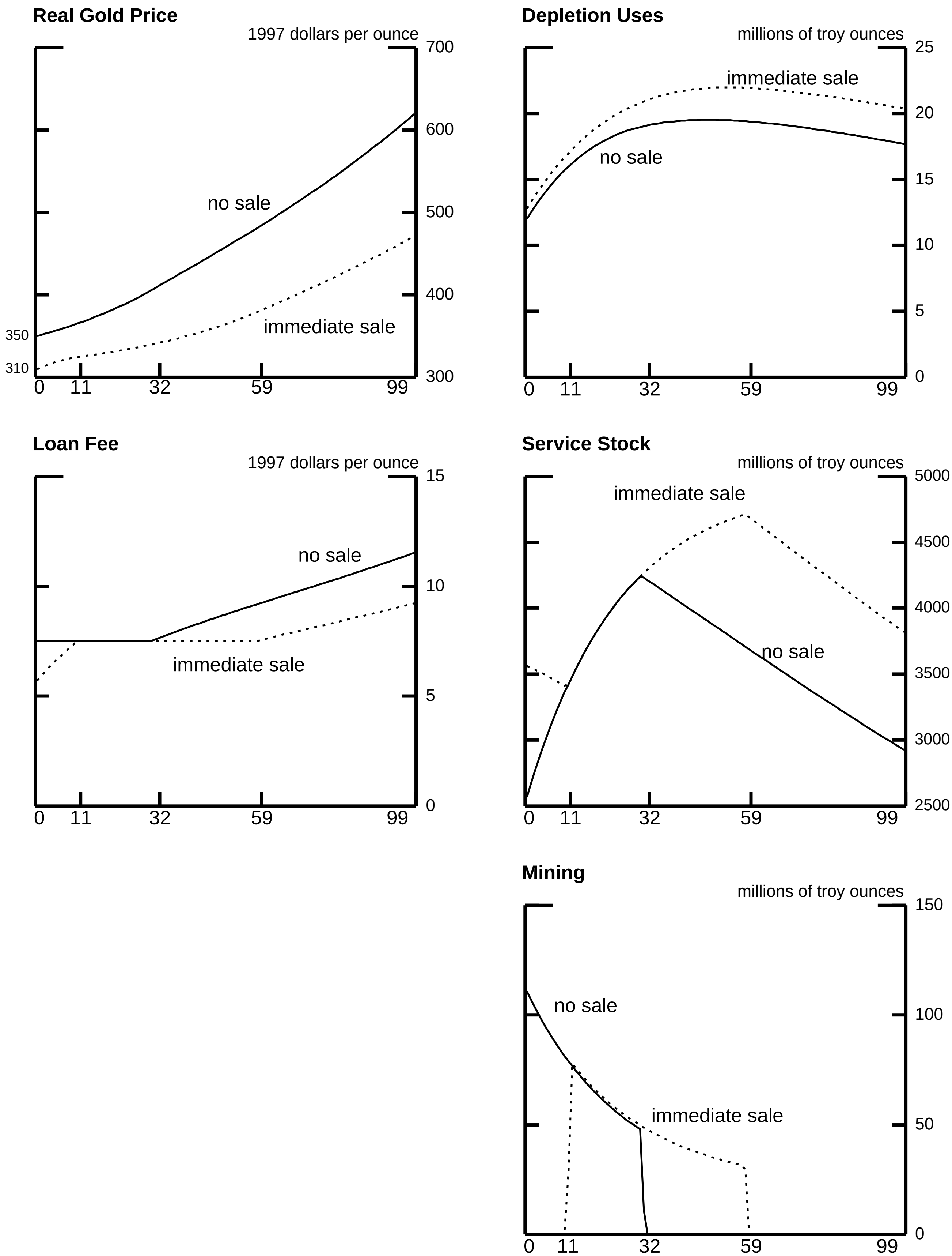


Figure 8

$\bar{H}$ < Reference $\bar{H}$

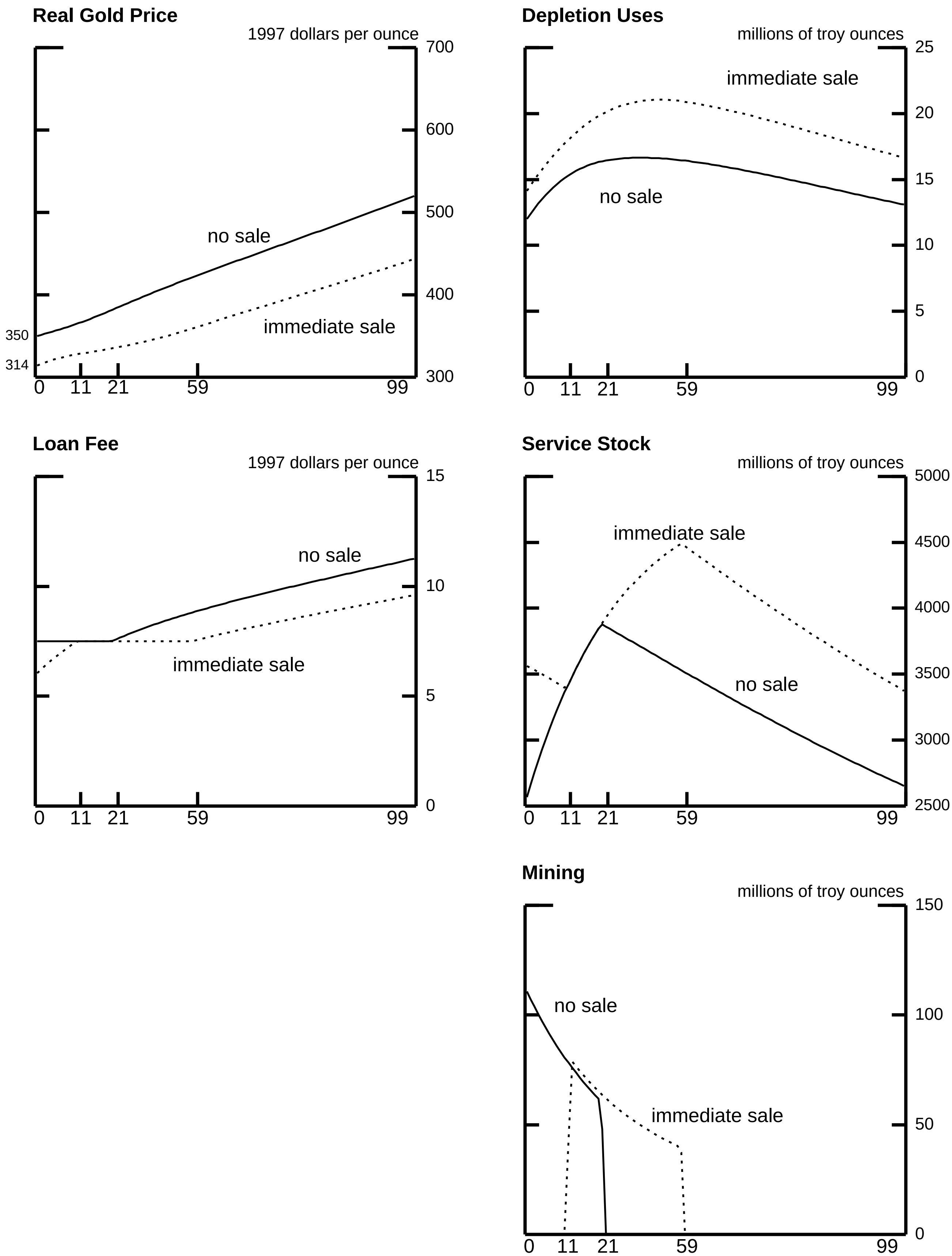


Figure 9

$\overline{H}$ > Reference $\overline{H}$

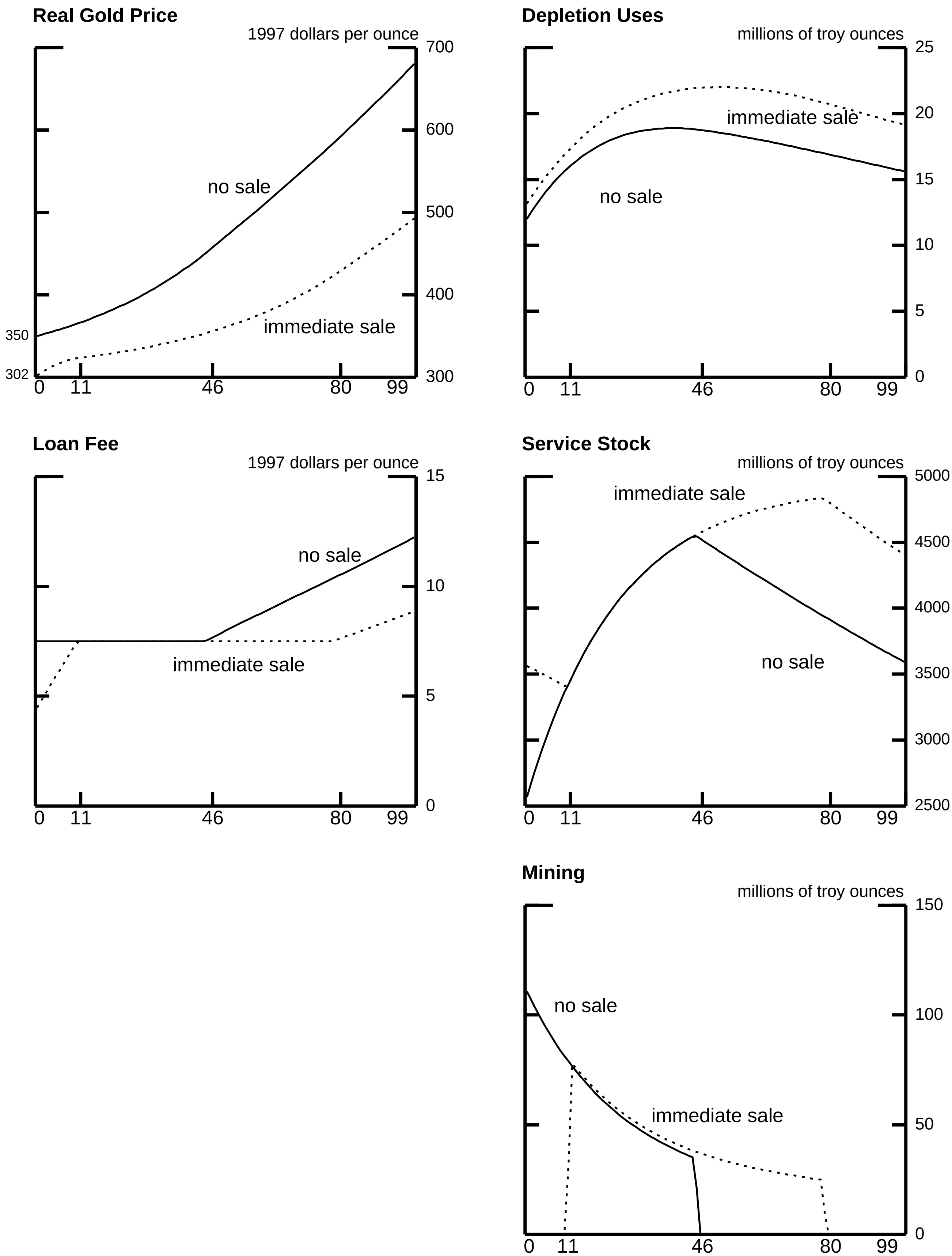
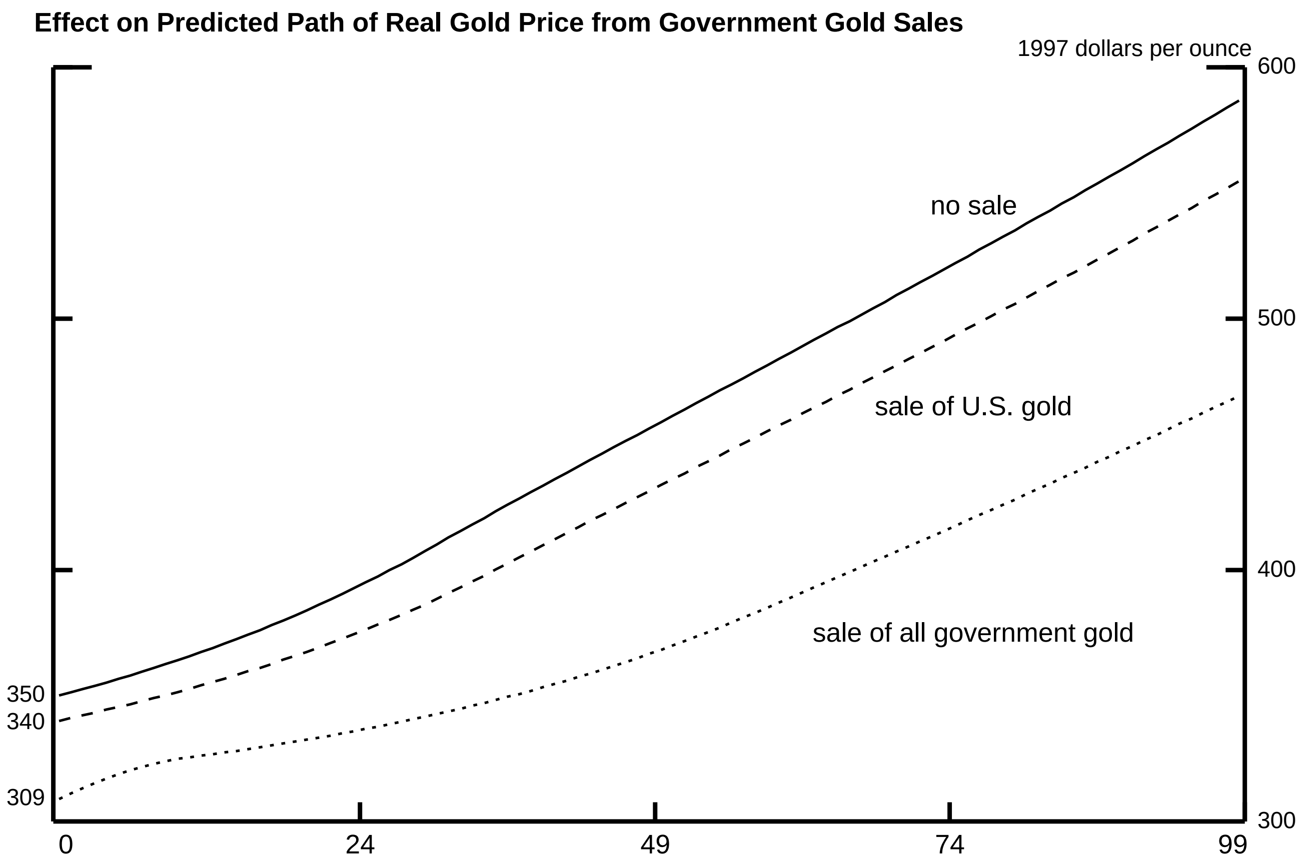


Figure 10



Effect on Predicted Path of Real Gold Price from Government Gold Sales
1997 dollars per ounce
600
500
400
300
350
340
309
no sale
sale of U.S. gold
sale of all government gold
0
24
49
74
99