Board of Governors of the Federal Reserve System

International Finance Discussion Papers

Number 785

December 2003

# Interest Rate Rules and Multiple Equilibria in the Small Open Economy

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Abstract: In a small open economy model with traded and non-traded goods this paper characterizes conditions under which interest rate rules induce aggregate instability by generating multiple equilibria. These conditions depend not only on how aggressively the rule responds to inflation, but also on the measure of inflation to which the government responds, on the degree of openness of the economy and on the degree of exchange rate pass-through. As an important policy implication, this paper finds that to avoid aggregate instability in the economy the government should implement an aggressive rule with respect to the inflation rate of the sector that has sticky prices. That is the non-traded goods inflation rate. As a by-product of this analysis, it is shown that "fear-of-floating" governments that follow a rule that responds to both the CPI-inflation rate and the nominal depreciation rate or governments that implement "super-inertial" interest rate smoothing rules may actually induce multiple equilibria in their economies.
This paper also shows that for forward-looking rules, the determinacy of equilibrium conditions depend not only on the degree of openness of the economy but also on the weight that the government puts on expected future CPI-inflation rates. In fact rules that are "excessively" forward-looking always lead to multiple equilibria.

Keywords: small open economy, multiple equilibria, interest rate rules, sticky prices and imperfect pass-through.

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# 1 Introduction

It has been argued that the only sound monetary policy for emerging economies is one based on the trinity of a flexible exchange rate, an inflation target and a monetary policy rule.[1] This argument has started a literature that has focused on studying the macroeconomic implications of implementing diverse monetary rules in the small open economy. Some examples of this literature are the works by Ball (1999), Clarida, Gali and Gertler (1998, 2001), Gali and Monacelli (2002), Kollmann (2002) and Svensson (2000) among others.[2]

Among diverse monetary rules this literature analyzes interest rate rules or Taylor rules whereby the government sets the nominal interest rate as an increasing function of inflation and the output gap.[3] In general terms these works follow two similar approaches. First, there are studies that address the question of which type of monetary rule a government should follow in order to minimize the variance of inflation and/or the variance of the output gap. Second, there are studies that propose a social welfare function and use it to rank these monetary rules.

As a by-product of this literature policy makers may find particular suggestions about the specifications of interest rate rules that increase welfare and reduce the variance of output and/or the variance of inflation in an economy that has been hit by different types of shocks. These suggestions focus not only on how aggressive the interest rate rules must be with respect to inflation but also on how exchange rates should be taken into account in the design of the rule for small open economies. As Svensson (2000) points out, this is a relevant issue since the exchange rate is a crucial channel of transmission of monetary policy into inflation in these economies.[4]

In these papers and in particular for the interest rate rules studied, conclusions are sensitive to the model specification, the chosen social welfare function and the type of shocks analyzed.[5] More importantly some of the models implicitly or explicitly assume parameters for the rules under

[1] See Taylor (2000) for instance.

[2] See also Ghironi (2000), Ghironi and Rebucci (2001), Lubik and Schorfheide (2003b) Monacelli (1999) and McCallum and Nelson (2001) among others.

[3] See Taylor (1993) and Henderson and McKibbin (1993).

[4] This channel arises because the exchange rate, and in particular, the nominal exchange rate depreciation may affect the price of traded goods, that in turn affects the Consumer Price Index (CPI). If there is a high pass-through, then currency depreciation might have a big impact on the CPI-inflation. Therefore if the government is interested in controlling the variability of the CPI-inflation, then it must find a way to avoid large swings in the nominal depreciation rate. One possible solution to this problem is to design an interest-rate feedback rule that, besides the CPI-inflation rate, also responds to the nominal depreciation rate.

[5] For instance, Svensson (2000) suggests that flexible CPI-inflation targeting and its derived optimal interest rate rule that includes the real exchange rate, stands out in limiting the variability of the CPI inflation, the output gap and the real exchange rate. On the other hand, Kollmann (2002), and Clarida et al. (2001) argue that under certain conditions, optimal monetary policy for the small open economy dictates that the central bank should follow the same interest rate feedback rule designed for the closed economy without taking into consideration the exchange rate.

analysis that always lead to a unique equilibrium. In this sense they do not take into account that multiple equilibria may arise in monetary business cycles models where the government follows an interest rate rule, a point that has been raised by Benhabib, Schmitt-Grohé and Uribe (2001a,b), Clarida, Gali and Gertler (2000), Carlstrom and Fuerst (2001,1999a), Dupor (2001) and Woodford (2002) for closed economies.[6]

Based on the previous observation, this paper follows a different approach to studying the implications of using interest rate rules in a small open economy. We pursue a determinacy of equilibrium analysis in order to isolate and identify conditions that are sufficient to ensure that these rules do not generate multiple equilibria in the aforementioned economy. Our objective is to answer the following question: *in the small open economy how can the government avoid aggregate instability due to multiple equilibria when it designs and follows an interest-rate feedback rule?* To answer this question, first we analyze simple interest rate rules that depend solely on a particular measure of inflation. Second, we study systematically how the inclusion of the nominal depreciation rate and other variables as arguments of the rule affects the determinacy of equilibrium; and third we focus on rules that, loosely speaking, include forward-looking and backward-looking elements.

We believe our approach is relevant for two reasons. First, we do not restrict the set of the parameters of the specification of the interest rate rule as the aforementioned literature implicitly does. Second, although it is not possible to determine if the rules that lead to multiple equilibria are welfare reducing, it is possible to show that they may generate fluctuations in the economy that are determined not only by fundamentals but also by self-fulfilling expectations. It is in this sense that these rules may generate aggregate instability in the economy, and policy makers may be interested in avoiding them.

The main contribution of this paper is to show that the conditions under which interest rate rules lead to multiple equilibria in the small open economy are not a simple extension of the conditions in closed economies.[7] In fact we show that some rules that in closed economies assure a unique equilibrium, in the small open economy may actually destabilize the economy by generating multiple equilibria (real indeterminacy).[8]

Previous works for closed economies have claimed that the type of rules that lead to multiple equilibria can be fully characterized by the magnitude of the interest rate response coefficient to

[6]It is important to point out that indeterminacy of the equilibrium (or multiple equilibria) may also arise under other type of rules such as money growth rules.

[7]See Benhabib, Schmitt-Grohé and Uribe (2001a), Clarida, Gali and Gertler (2000), Dupor (2001) and Woodford (2002) among others, for closed economy analyses.

[8]We will use the terms multiple equilibria and real indeterminacy interchangeably. In fact the type of indeterminacy of equilibrium that we deal with in this paper corresponds to real indeterminacy instead of nominal indeterminacy. We say that the equilibrium displays real indeterminacy if there exists an infinite number of equilibrium sequences of inflation and real variables of the model such as consumption.

inflation. If this coefficient is greater than one then the rule is considered an active one and it implies that the government aggressively fights inflation by raising the nominal interest rate by more than the increase in inflation. On the other hand, if this coefficient is less than one then the rule is considered a passive one which means that the government underreacts to inflation by raising the nominal interest rate by less than the increase in inflation. These previous works have also suggested that in order to stabilize the closed economy and avoid multiple equilibria the government should follow only active rules.[9] Our analysis shows that this claim does not necessarily hold in the small open economy. We show that conditions under which interest rate rules lead to multiple equilibria depend not only on the type of monetary policy, active or passive, but also on the measure of inflation to which the government responds, on the degree of openness of the economy and on the degree of exchange rate pass-through.

With respect to the measure of inflation to which the government responds, we find the following. Under perfect exchange rate pass-through the traded goods inflation rate coincides with the nominal depreciation rate, and a rule whose sole argument is the nominal depreciation rate always leads to multiple equilibria regardless of how active or passive the rule is. The intuition of this result can be constructed taking into account the following features of the model. First since the rule responds solely to the nominal depreciation rate then the government does not react to people's expectations about the non-traded goods inflation. Second the evolution consumption of non-traded goods is determined by the real interest rate defined as the difference between the nominal interest rate, maneuvered by the government, and the expected non-traded goods inflation rate. Third, firms in the non-traded sector set their prices. The intuitive argument is based on constructing a self-fulfilling equilibrium as follows. Assume that people expect a higher non-traded goods inflation. Since the government does not react to these expectations then the real interest rate in terms of the expected non-traded goods inflation rate will decrease. This will increase consumption of non-traded goods to which firms will respond increasing prices of non-traded goods validating the people's original expectations.

In contrast, if the only argument of the rule is the non-traded goods inflation rate, multiple equilibria arise solely under passive rules whereas equilibrium uniqueness is guaranteed by active rules.[10] The reason is that under active (passive) rules if people expect a higher non-traded goods inflation then the government reacts to these expectations increasing (decreasing) the real interest rate in terms of the expected non-traded goods inflation rate. This will decrease (increase) consumption of non-traded goods to which firms will respond decreasing (increasing) prices of non-traded goods destroying (validating) the people's original expectations.

[9] See Clarida et al. (2000).

[10] In our model we assume that the prices of the traded goods are flexible while the prices of the non-traded goods are sticky. This assumption plays an important role in our results. More on this below.

To the extent that the CPI-inflation is a weighted average of the traded goods inflation rate and the non-traded goods inflation rate, the previous results imply that an active rule whose sole argument is the CPI-inflation may lead to real indeterminacy. Moreover these results suggest that governments in small open economies should design rules satisfying two requirements. First, the measure of inflation of the rule should be the non-traded goods inflation or at least a measure of inflation that is not heavily affected by the nominal depreciation rate. Second, the rule should be active. Interestingly this suggestion coincides with some of the proposals of the aforementioned literature. In particular Clarida, Gali and Gertler (2001) and Kollmann (2002) emphasize that under perfect exchange rate pass-through, optimal monetary policy calls for a government that targets the domestic inflation instead of the CPI-inflation, making it the measure of inflation of the rule.[11] However it is important to notice that these works arrive at these conclusions *without* pursuing a determinacy of equilibrium analysis as we do in the present paper.

Furthermore in our model the measure of openness of the economy corresponds to the share of traded goods. Since the CPI-inflation is a weighted average of the traded goods inflation rate and the non-traded goods inflation rate, where the weights are related to the share of traded goods, it is understandable that the determinacy of equilibrium conditions also depend on the degree of openness of the economy. To understand this, note that the more open the economy is the more similar the CPI-inflation rate and the traded goods inflation rate (nominal depreciation rate) become. On the other hand the more closed the economy is the more similar the CPI-inflation rate and the non-traded good inflation become. Using this and the previous results for interest rate rules, it is possible to infer that an active rule that responds to the CPI-inflation rate may lead to multiple equilibria if the economy is very open. In contrast the same active rule guarantees a unique equilibrium if the economy is very closed. In fact what we find is that the more open the economy is the more likely it is that an active rule will lead the economy to multiple equilibria. These results call into question the interpretation given to some of the estimations of interest rate rules in small open economies. In particular, empirical works like Clarida et al. (1998) have claimed that active interest rate rules are preferable since they induce stability in inflation and in the whole economy. Our results imply that this claim is not necessarily valid since the conditions that determine whether a rule leads to instability depend not only on the interest rate response coefficient to inflation but also on the degree of openness of the economy.

Introducing imperfect exchange rate pass-through in the model does not change the basic results. That is, although the determinacy of equilibrium depends on the degree of exchange rate pass-through, we find that under a high exchange rate pass-through, the most suitable policy for the government to avoid inducing aggregate instability in the economy is to target the non-traded

[11] A similar proposal by Ball (1999) points out the importance of targeting a modified inflation index that filters out the transitory effects of exchange rate movements, or to use an average of CPI-inflation over a longer period.

goods inflation rate. Devereux and Lane (2001) have similar proposals in the sense that they point out that when there is a high exchange rate pass-through, a policy of non-traded goods inflation targeting does better stabilizing the economy and in terms of welfare than a policy of CPI-inflation targeting.

We also study more general interest rate rules. In these rules, the interest rate may respond not only to a measure of inflation but also to other variables such as the output gap, the nominal depreciation rate, the real exchange rate, or the weighted average of past interest rates. For these rules we also find that depending on the degree of openness, active rules with respect to the CPI-inflation rate may induce multiple equilibria. This result holds independently of how big the positive interest rate response is with respect to the other arguments. As a by-product of this analysis we find that "fear of floating" governments that follow a rule that responds to both the CPI-inflation rate and the nominal depreciation rate may induce aggregate instability in their economies. And that even "super-inertial" smoothing interest rate rules may lead to multiple equilibria when the economy is very open. This result contrasts with some results in the closed economy literature. In particular Rotemberg and Woodford (19999) and Giannoni and Woodford (2002) have shown that rules with an interest rate smoothing coefficient that is greater than one guarantee a locally unique equilibrium and are, in addition, capable of implementing the optimal real allocation.

Finally we study rules that depend on either the weighted average of expected future CPI-inflation rates or on the weighted average of past CPI-inflation rates. Under the former we show that the determinacy of equilibrium conditions depend not only on the degree of openness but also on the weight the monetary authority puts on expected future CPI-inflation rates. If the central bank puts an "excessively" high weight on distant expected future CPI-inflation rates then the rules always lead to multiple equilibria. On the other hand, backward-looking interest-rate rules always lead to a unique equilibrium if the rule is active with respect to the weighted average of past CPI-inflation rates.

In the open economy literature there are papers that pursue a determinacy of equilibrium analysis of interest rate rules using models with two similar countries (Benigno and Benigno (2000) and Benigno, Benigno and Ghironi (2000)). However in these works the degrees of openness of the economies do not play any role for the determinacy of equilibrium since they focus on rules whose measure of inflation corresponds to the price inflation of the goods that each country produces (domestic inflation).[12]

Carlstrom and Fuerst (1999b) consider a limited participation model for the small open economy with one good to pursue a determinacy of equilibrium analysis. They assume flexible prices and study backward and forward-looking interest-rate rules. Since they only consider one good, the degree of openness of the economy and the measure of inflation to which the government responds

[12] In Benigno, Benigno and Ghironi (2000), they focus on rules that react exclusively to the nominal exchange rate.

do not play any role in their results.[13]

The remainder of this paper is organized as follows. Section 2 presents the set-up of the model with its main assumptions. Section 3 pursues the determinacy analysis for different interest-rate rule specifications. Finally Section 4 concludes.

# 2 The Model

## 2.1 The Household-Firm Unit

Consider a small open economy inhabited by a large number of identical individuals blessed with perfect foresight. The individuals live infinitely and the preferences of the representative agent can be described by the intertemporal utility function[14]

$$U_0 = \int_0^{\infty} \left[ A(c_{Tt}, c_{Nt}) + (1 - h_{Tt} - h_{Nt}(j)) + \chi \log(m_t) - \frac{\gamma}{2} \left( \frac{\dot{P}_{Nt}(j)}{P_{Nt}(j)} - \pi_N^{ss} \right)^2 \right] \mathrm{e}^{-\beta t} dt \tag{1}$$

$$A(c_{Tt}, c_{Nt}) = \alpha \log(c_{Tt}) + (1 - \alpha) \log(c_{Nt}) \tag{2}$$

where $\alpha,\ \beta \in (0,1)$, and $\gamma\ ,\ \chi > 0$; $c_{Tt}$ and $c_{Nt}$ denote the consumption of traded and non-traded goods respectively, $h_{Tt}$ and $h_{Nt}(j)$ are the labor allocated to the production of the traded good and to the $jth$ variety of the non-traded good respectively and $m_t$ refers to real money holdings $\left(\frac{M_t}{E_t}\right)$. Equations (1) and (2) imply that the representative individual derives utility from consuming traded and non-traded goods, from not working in either sector and from the liquidity services of money.

In order to understand the last term of equation (1) we assume that besides producing the traded good, the representative household-firm unit also produces the $jth$ variety of non-traded good. The production process only requires labor and makes use of following instantaneous production technologies

$$y_{Tt} = (h_{Tt})^{\theta_T} \qquad and \qquad y_{Nt}(j) = (h_{Nt}(j))^{\theta_N} \tag{3}$$

[13] In the process of completing the first version of the present paper we became aware of independent works by Linnemann and Schabert (2002) and De Fiore and Zheng (2003). These works as the present paper find that the degree of openness matters for the determinacy of equilibrium analysis for interest rate rules in the small open economy. In the present paper we show that the determinacy of equilibrium results associated with the degree openness are linked to the results associated with the measure of the inflation to which the government responds. The reason is that in our model, it is the effect of the nominal depreciation rate on the CPI-inflation rate what drives some of the real indeterminacy results for the interest rate rules under study. Furthermore we also discuss the importance of the degree of exchange rate pass-through in the aforementioned analysis.

[14] We use specific functional forms since this will simplify the analysis, allowing us to convey the main message of the paper. In the last part of the paper we discuss how our main results still hold for a utility function that considers an elasticity of substitution between traded and non-traded goods and an intertemporal elasticity of substitution different than one.

where $0 < \theta_T < 1$ and $0 < \theta_N < 1$. In the non-traded good production, it is assumed that the representative agent is also subject to the constraint that, given the price she charges $P_{Nt}(j)$ for the $jth$ variety, her sales are demand determined. This demand constraint can be derived using Dixit-Stiglitz preferences over differentiated goods. It usually takes the following form

$$y_{Nt}(j) = Y_t^d \left( \frac{P_{Nt}(j)}{P_{Nt}} \right)^{\phi} \tag{4}$$

where $P_{Nt}$ is the economy-wide price level of the non-traded goods, $Y_t^d$ is the aggregate demand for non-traded goods and $\phi < -1$.

Under this production framework, the last term of the intertemporal utility function means that the household-firm unit derives utility from hitting a target of own non-traded price change $\left(\frac{\dot{P}_{Nt}(j)}{P_{Nt}(j)}\right)$ of the $jth$ variety. This approach to model the cost of nominal price adjustment is due to Rotemberg (1982) and it is a simple way to introduce price-stickiness in the model.[15] It basically implies that households dislike having their price of non-traded goods of the $jth$ variety grow at a rate different from the steady-state non-traded good inflation rate, $\pi_N^{ss}$.[16] Moreover since most of the works of the aforementioned literature of monetary rules in the small open economies introduce sticky-prices, we will also assume this type of distortion to make our results and theirs comparable.

It is also assumed that the law of one price holds for the traded good and to simplify the analysis we normalize the foreign price of the traded good to one. Therefore, the domestic currency price of the traded good ($P_{Tt}$) is equal to the nominal exchange rate ($E_t$). That is $P_{Tt} = E_t$. This simplification in tandem with the preferences aggregator described by equation (2) can be used to derive the consumer price index (CPI),[17]

$$p_t = \frac{(E_t)^{\alpha} (P_{Nt})^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \tag{5}$$

Using equation (5) and defining the nominal devaluation rate as $\epsilon_t = \dot{E}_t/E_t$, it is straightforward to derive the CPI inflation rate, $\pi_t$, as a weighted average of the nominal depreciation rate, $\epsilon_t$, and the inflation of the non-traded goods, $\pi_{Nt} = \dot{P}_{Nt}/P_{Nt}$, that is

$$\pi_t = \alpha\epsilon_t + (1-\alpha)\pi_{Nt} \tag{6}$$

It is important to notice that the weights in equation (6) depend on the share of traded goods, $\alpha$. This share corresponds to a measure of "openness" of the economy in the present model.

[15] Benhabib et al. (2001a,b) and Dupor (2001) also follow this approach to model price stickiness. An alternative approach follows Calvo (1983) and Yun (1996).

[16] The superscript "$ss$" refers to the steady state.

[17] To derive this equation see the Theory of Price Indeces in Obstfeld and Rogoff (1996).

We suppose that this economy follows a flexible exchange rate regime and we define the real exchange rate ($e_t$) as the ratio between the price of traded goods ($E_t$) and the aggregate price of non-traded goods ($P_{Nt}$),

$$e_t = E_t/P_{Nt} \tag{7}$$

From the definition of the real exchange rate in (7) it is straightforward to deduce that

$$\frac{\dot{e}_t}{e_t} = \epsilon_t - \pi_{Nt} \tag{8}$$

Moreover we assume that the representative household-firm unit can invest in two types of interest-bearing assets: domestic bonds issued by the government, $A_t$, that pay a nominal interest rate, $R_t$; and foreign currency denominated bonds, $b_t$, that pay a constant interest rate, $r$. The real values of these assets will be denoted by $a_t = A_t/E_t$ and $b_t$, respectively.

We introduce portfolio adjustment costs for the foreign bonds assuming the following functional form

$$z_t = \frac{\psi}{2}\left(b_t - b^{ss}\right)^2 \tag{9}$$

where $\psi > 0$, is a parameter that measures the degree of capital mobility and $b^{ss}$ represents the steady state of the stock of foreign bonds. We introduce these costs to solve the "unit root problem" in discrete time models or the "zero root problem" in continuous time models of a small open economy. Such a problem arises due to the popular assumption in International Macroeconomics that the subjective rate of discount ($\beta$) is constant and equal to the international interest rate ($r$). This assumption introduces a random walk in equilibrium consumption making the steady state dependent on the initial stock of wealth. As a result, the presence of a zero root in a dynamic system implies that it is not valid to apply the usual technique of linearizing the system around the steady state and studying the eigenvalues of the Jacobian matrix in order to characterize local determinacy.[18] Given that we are particularly interested in pursuing a local equilibrium determinacy analysis for interest rate rules, we introduce convex portfolio adjustment costs for foreign bonds, as described by (9). This approach can be considered as one that assumes incomplete markets and it is one of the possible solutions that Schmitt-Grohé and Uribe (2003) analyze to solve the "unit root problem" in the small open economy.

In addition following the approaches of assuming complete markets or using an elastic-interest rate premium ($r_t = r^* + \psi(b_t - b^{ss})$, where $r^*$ is the free-risk international interest rate and $\psi(b_t - b^{ss})$

[18] The basic problem is that the possibility of studying a nonlinear differential equation system using its linearized version relies on the "Theorem of Hartman and Grobman" (see Guckenheimer and Holmes (1983)). However if the Jacobian matrix has a zero eigenvalue it is not clear that one can draw conclusions about the nonlinear system applying this theorem and using the linearized version of the system. See Giavazzi and Wyplosz (1985).

corresponds to the risk premium) will not affect the results of this paper. [19]

The representative agent's instantaneous budget constraint can be written as follows

$$\dot{b}_t = rb_t + R_t a_t + \tau_t + y_{Tt} + \frac{P_{N_t}(j)}{P_{N_t}} \frac{y_{Nt}(j)}{e_t} - c_{T_t} - \frac{c_{Nt}}{e_t} - \epsilon_t (m_t + a_t) - (\dot{m}_t + \dot{a}_t) - z_t \tag{10}$$

where $\tau_t$ denotes lump-sum transfers from the government. Equation (10) says that the accumulation of foreign bonds is equal to the difference of the agent's disposable income and her expenditures. Her income is determined by the interests received by all kind of bonds, the transfers from the government, and her income from producing the traded good and the $jth$ variety of the non-traded good. Her expenditures consist of consumption of traded and non-traded goods, her holdings of money and domestic bond balances, eroded by domestic currency depreciation, and the convex portfolio adjustment costs.

Finally the representative Household-Firm unit is also subject to an Non-Ponzi game condition of the form

$$\lim_{t \to \infty} \mathsf{e}^{-\beta t} (m_t + a_t + b_t) \geq 0$$

The problem of the agent is reduced to choose $c_{T_t}$, $c_{Nt}$, $h_{Tt}$, $h_{Nt}(j)$, $m_t$, $a_t$, $b_t$ and $P_{Nt}(j)$ in order to maximize (1) subject to (2), (3), (4), (9), (10) and the Non-Ponzi game condition, given $b_0$, $a_0$, $m_{0,}$ $P_{N0}(j)$, $b^{ss}$, $\pi_N^{ss}$ and the time paths for $r$, $R_t$, $\epsilon_t$, $Y_t^d$, $P_{Nt}$ and $\tau_t$.

The first order conditions associated with this optimization problem can be written as[20]

$$\frac{\alpha}{c_T} = \lambda \tag{11}$$

$$\frac{1-\alpha}{c_N} = \frac{\lambda}{e} \tag{12}$$

$$\theta_T h_T^{(\theta_T - 1)} = \frac{1}{\lambda} \tag{13}$$

$$\left( \frac{P_N(j)}{P_N} \frac{1}{e} - \frac{\mu}{\lambda} \right) \theta_N \left( h_N(j) \right)^{(\theta_N - 1)} = \frac{1}{\lambda} \tag{14}$$

$$\frac{\chi}{m} - \lambda \epsilon = -\dot{\lambda} + \lambda \beta \tag{15}$$

[19] These analyses are available from the author upon request. The approach of complete markets was not used since there are works that have found evidence against it in open economies. See Kollmann (1995) among others.

[20] For simplicity from now on we ignore the time subscript "$t$".

$$\lambda(R-\epsilon)=-\dot{\lambda}+\lambda\beta \tag{16}$$

$$\lambda(r-\psi\left(b-b^{ss}\right))=-\dot{\lambda}+\lambda\beta \tag{17}$$

$$\dot{\pi}_N(j)=r(\pi_N(j)-\pi_N^{ss})-\frac{\lambda}{\gamma}\frac{P_N(j)}{P_N}\frac{h_N^{\theta_N}(j)}{e}-\frac{\mu\phi}{\gamma}\frac{P_N(j)}{P_N}Y^d\left(\frac{P_N(j)}{P_N}\right)^{\phi-1} \tag{18}$$

$$\lim_{t\to\infty}\mathsf{e}^{-\beta t}(m+a+b)=0 \tag{19}$$

where $\lambda$ is the co-state variable or in economic terms the shadow price of wealth, $\mu$ is the multiplier associated with the demand constraint (4) and $\pi_N(j)=\frac{\dot{P}_N(j)}{P_N(j)}$.

From now on we will focus on a symmetric equilibrium for which $P_N(j)=P_N$ and $h_N(j)=h_N$. We proceed giving an interpretation to the first order conditions. Equation (11) equates the marginal utility of traded goods to the marginal utility of wealth. Combining equations (11) and (12) we obtain

$$\frac{\alpha c_N}{(1-\alpha)c_T}=e \tag{20}$$

implying that the marginal rate of substitution between traded and non-traded goods is equal to their relative price or real exchange rate.

From equations (13) and (14) and imposing equilibrium symmetry it is possible to derive

$$\left(1-\frac{\mu e}{\lambda}\right)\theta_N\left(h_N\right)^{(\theta_N-1)}=e\theta_T h_T^{(\theta_T-1)}$$

that equalizes the marginal revenue products of labor among sectors.

From equations (11), (15) and (16) we can deduce the demand for real balances of money as an increasing function of the consumption of traded goods and a decreasing function of the nominal interest rate of the domestic bonds offered by the government. That is,

$$m=\frac{\chi}{\alpha}\frac{c_T}{R} \tag{21}$$

Moreover as a consequence of the introduction of convex portfolio adjustment costs for the foreign assets, the typical Uncovered Interest Parity (UIP) condition does not hold in this model. To see this, we can use equations (16) and (17) to derive the following expression

$$r+\epsilon-\psi(b-b^{ss})=R \tag{22}$$

This is a revised version of the UIP condition. It is still an arbitrage condition that equalizes the returns from the domestic bond and the foreign bond. However what makes it different from the typical UIP condition is that the return to foreign bonds includes the marginal cost of adjusting the portfolio of foreign bonds $\psi(b - b^{ss})$. It should be observed that the demand for foreign bonds is not indeterminate in this model. More precisely equation (22) allows us to find a net demand for foreign bonds. That is,

$$b - b^{ss} = \frac{1}{\psi}(r + \epsilon - R) \tag{23}$$

where the parameter $\psi$ can be used to parameterize the degree of capital mobility. In the case of perfect capital mobility, $\psi \to 0$, and equation (22) reduces to the typical UIP condition $r + \epsilon = R$. In the case of zero capital mobility $\psi \to \infty$ and $b = b^{ss}$, i.e domestic residents always hold a constant stock of foreign bonds and cannot adjust their portfolio using this type of assets.

Using equation (22), the typical Euler Equation for the shadow price of wealth (16) can be rewritten as

$$\frac{\dot{\lambda}}{\lambda} = \psi\left(b - b^{ss}\right) \tag{24}$$

Similarly using (8), (11), (16) and (20) together with the assumption $\beta = r$, we can derive an Euler equation for the consumption of non-traded goods,[21]

$$\frac{\dot{c}_N}{c_N} = R - \pi_N - r \tag{25}$$

Finally, we can utilize equations (12), (14) and (18), and the equilibrium symmetry conditions $P_N(j) = P_N$ , $h_N(j) = h_N$ , $\pi_N(j) = \pi_N(j)$, altogether with the equilibrium condition for the non-traded good ($y_N = h_N^{\theta_N} = c_N$), to derive the following differential equation for the non-traded goods inflation

$$\dot{\pi}_N = r(\pi_N - \pi_N^{ss}) - \frac{(1+\phi)(1-\alpha)}{\gamma} + \frac{\phi}{\gamma\theta_N}(c_N)^{\frac{1}{\theta_N}} \tag{26}$$

where $\phi < -1$. This equation corresponds to a new Phillips equation for non-traded goods inflation. We can establish a relationship between this equation derived using sticky prices "*a la*" Rotemberg (1982) and a similar equation that we would have derived if we had introduced price stickiness following the approach of Calvo (1983). To simplify the comparison assume that it is possible to have $\theta_N = 1$. Therefore using the fact that in equilibrium $y_N = h_N = c_N$, equation (26) can be written as

[21] Note that in this model the assumption $\beta = r$ does not casue the zero root problem, since $\frac{\dot{\lambda}}{\lambda} = \psi\left(b - b^{ss}\right)$.

$$\dot{\pi}_N = r(\pi_N - \pi_N^{ss}) + \frac{\phi}{\gamma}(c_N - y_N^{ss}) \tag{27}$$

where $y_N^{ss} = \frac{(1-\alpha)(1+\phi)}{\phi}$. It should be noticed that the last term of equation (27) can be seen as a measure of excess demand in the non-traded goods market. Therefore as in Calvo (1983), the change of the non-traded goods inflation rate is a negative function of the excess of demand, given that $\phi < -1$. Furthermore if we assume that $\pi_N^{ss} = 0$ and iterate forward (27) we obtain

$$\pi_{Nt} = \int_t^{\infty} \mathrm{e}^{-r(s-t)} \frac{-\phi}{\gamma}(c_{Ns} - y_N^{ss})\, ds \tag{28}$$

that implies that if at time $t$, the actual excess of demand is expected to be positive, then firms increase prices because the demand for non-traded goods is "too high". A larger $\gamma$ implies that adjustment costs of prices are higher and therefore the non-traded goods inflation responds less to the excess of demand of these goods.

## 2.2 The Government

We will assume that the government issues two nominal liabilities: money, $M^g$, and a domestic bond, $A^g$, that pays a nominal interest rate $R$. The real values of these nominal variables are denoted by $m^g$ and $a^g$, respectively. It is assumed the government makes lump-sum transfers to households, $\tau$, and pays interest on its debt ($Ra^g$). Moreover it receives revenues from seigniorage $\left(\frac{\dot{M}^g + \dot{A}^g}{E} = \epsilon(m^g + a^g) + \dot{m}^g + \dot{a}^g\right)$. The government has no access to foreign bonds. This assumption simplifies the model and it does not have serious implications for our results if the interest is in analyzing cases in which the exchange rate is flexible.

Under these assumptions the government budget constraint can be written as follows

$$\dot{m}^g + \dot{a}^g = Ra^g + \tau - \epsilon(m^g + a^g) \tag{29}$$

The fiscal policy is Ricardian. That is the government picks the path of $\tau$ satisfying the intertemporal version of (29) in conjunction with the transversality condition,

$$\lim_{t\to\infty}(m^g + a^g)\exp\left(-\int_0^t (R-\epsilon)ds\right) = 0 \tag{30}$$

Finally we will define the monetary policy as an interest-rate feedback rule whereby the government sets the nominal interest rate as an increasing function of one or more variables. The possible variables are: the contemporaneous CPI-inflation rate ($\pi$), the nominal depreciation rate ($\epsilon$), the non-traded goods inflation ($\pi_N$), the real exchange rate ($e$), the real depreciation rate ($\dot{e}/e$), and output ($y$),

$$
\begin{aligned}
R &= \rho(\pi, \epsilon, e, \frac{\dot{\mathrm{e}}}{\mathrm{e}}, y) && (31)\\
&= R^{ss} + \rho_\pi(\pi - \pi^{ss}) + \rho_\epsilon(\epsilon - \epsilon^{ss}) + \rho_{\pi_N}(\pi_N - \pi^{ss}) + \rho_e(e - e^{ss}) + \rho_{\dot{e}/e}(\dot{e}/e) + \rho_y(y - y^{ss})
\end{aligned}
$$

where as can be seen $\rho(.)$ is continuous and non-decreasing in $\pi$, $\epsilon$, $\pi_N$, $e$, $\dot{e}/e$, and $y$.[22]

To understand the measure of output in the interest-rate feedback rule we provide a definition. We define output in this economy under equilibrium symmetry ($y_N(j) = y_N$) as the sum of the production in both sectors, traded ($y_T$) and non-traded ($y_N$), valued at the real prices $q_T = \frac{P_T}{p} = \frac{E}{p}$ and $q_N = \frac{P_N}{p}$; thus[23]

$$
y = q_T y_T + q_N y_N \tag{32}
$$

To complete the characterization of the feedback interest rate rule we make another assumption and provide a definition based on Leeper (1991). We assume that there exist one $\pi^{ss} = \epsilon^{ss} = \pi_N^{ss} > -r$ , and $e^{ss}$, $y^{ss}$ such that in steady state $R^{ss} = \rho(\pi^{ss}, e^{ss}, y^{ss}, R^{ss}) = r + \pi^{ss}$.

**Definition 1** *An interest-rate rule $R = \rho(x, w)$ is active (passive) with respect to $x$ or in terms of $x$, if $\frac{\partial R}{\partial x} = \rho_x > 1$ ($\frac{\partial R}{\partial x} = \rho_x < 1$).*

## 2.3 The Current Account

In order to derive the flow constraint for this economy we recall the equilibrium symmetry conditions $P_N(j) = P_N$ , $h_N(j) = h_N$, and the equilibrium conditions for the non-traded good market, $y_N = h_N^{\theta_N} = c_N$, the money market, $m = m^g$, and the domestic bond market, $a = a^g$. Then, we add equations (10) and (29) obtaining

$$
\dot{b} = rb + y_T - c_T - z \tag{33}
$$

that describes the evolution of the current account. Note that the portfolio adjustment costs $z_t = \frac{\psi}{2}\left(b_t - b^{ss}\right)^2$, appear in this equation as a cost for the whole economy.

## 2.4 A Perfect Foresight Equilibrium

To give the definition of the perfect foresight equilibrium in this model, we can simplify the expressions for the rule in (31), for the current account in (33) and for the revised UIP condition in (22) as follows.

[22] Note that in continuous-time an inflation rate is associated to the right-hand side derivative of a price with respect to time. This means that the type of rule we are analyzing has a forward-looking flavor. However this feature of the model agrees with the empirical findings of Clarida et al. (1998).

[23] Remember that $p$ is the CPI-price index.

Using the equilibrium condition for the non-traded good ($y_N = h_N^{\theta_N} = c_N$), and equations (3), (5), (6), (7), (12) and (13) we can rewrite the interest rate rule in (31) as

$$\begin{aligned} R &= \rho(\pi, \epsilon(\pi, \pi_N), \pi_N, e(\lambda, c_N), g(\pi, \pi_N), y(\lambda, c_N)) \\ &= \check{\rho}(\pi, \pi_N, \lambda, c_N) \end{aligned} \tag{34}$$

In addition using equations (11) and (13) we can rewrite the current account equation in (33) as

$$\dot{b} = rb + \left(\frac{1}{\lambda\theta_T}\right)^{\frac{\theta_T}{\theta_T - 1}} - \frac{\alpha}{\lambda} - \frac{\psi}{2}\left(b_t - b^{ss}\right)^2 \tag{35}$$

Finally utilizing equations (6) and (22) we can rewrite the revised UIP condition as

$$\psi(b - b^{ss}) = r + \frac{\pi - (1-\alpha)\pi_N}{\alpha} - R \tag{36}$$

**Definition 2** *Given $b_0$, $\pi_N^{ss}$ and $b^{ss}$ and under the assumption that the fiscal regime is Ricardian, a Perfect Foresight and Symmetric Equilibrium is defined as a set of sequences $\{\lambda, b, c_N, \pi_N, \pi, R\}$ satisfying:*

***a)*** *The Euler equation for the shadow price of wealth, equation (24).*

***b)*** *The Current Account equation (35).*

***c)*** *The Euler equation for consumption of non-traded goods, equation (25).*

***d)*** *The new Phillips equation for non-traded goods inflation, equation (26).*

***e)*** *The revised UIP condition, equation (36).*

***f)*** *The interest rate feedback rule, equation (34).*

Given the equilibrium set of sequences $\{\lambda, b, c_N, \pi_N, \pi, R\}$ then the corresponding sequences $\{\epsilon\}$, $\{c_T\}$, $\{e\}$, $\{a\}$, $\{m\}$, $\{h_T\}$ and $\{h_N\}$, are uniquely determined by equations (6), (10), (11), (13), the transversality condition (19), equations (20), (21), and the equilibrium condition for non-traded goods.

# 3 The Determinacy of Equilibrium Analysis

In order to accomplish the determinacy of equilibrium analysis we reduce the model as follows. From the interest rate rule in (34) and from the revised UIP condition (36) we can implicitly solve for the CPI-inflation, $\pi$, in terms of $b$, $\pi_N$, $\lambda$, and $c_N$; that is[24]

$$\pi = \pi\left(b, \pi_N, \lambda, c_N\right) \tag{37}$$

[24] Note that $r$ and $b^{ss}$ are considered constants.

Then we can substitute this expression into (34) to obtain

$$\begin{aligned} R &= \check{\rho}\left(\pi\left(b, \pi_N, \lambda, c_N\right), \pi_N, \lambda, c_N\right) \\ &= \hat{\rho}\left(b, \pi_N, \lambda, c_N\right) \end{aligned} \tag{38}$$

and finally we replace equations (37) and (38) into equation (25) to derive the following differential equation

$$\frac{\dot{c}_N}{c_N} = \hat{\rho}\left(b,\ \pi_N,\ \lambda,\ c_N,\right) - r - \pi_N \tag{39}$$

that together with

$$\dot{\pi}_N = r(\pi_N - \pi_N^{ss}) - \frac{(1+\phi)(1-\alpha)}{\gamma} + \frac{\phi}{\gamma\theta_N}(c_N)^{\frac{1}{\theta_N}} \tag{40}$$

$$\frac{\dot{\lambda}}{\lambda} = \psi\left(b - b^{ss}\right) \tag{41}$$

$$\dot{b} = rb + \left(\frac{1}{\lambda\theta_T}\right)^{\frac{\theta_T}{\theta_T - 1}} - \frac{\alpha}{\lambda} - \frac{\psi}{2}\left(b_t - b^{ss}\right)^2 \tag{42}$$

help us to characterize the equilibrium of this economy. The proof of existence of a steady-state equilibrium in this system is straightforward.

Although this is a system of four differential equations in four variables $\lambda$, $b$, $\pi_N$, and $c_N$, it can be easily analyzed qualitatively after one realizes that equations (41) and (42) do not depend on the inflation rate of non-traded goods ($\pi_N$) and the consumption of non-traded goods ($c_N$). Therefore these two equations form a system of two differential equations in two variables: the shadow price of wealth ($\lambda$) and the stock of foreign bonds ($b$). *This is an important consequence of the assumption that the utility function is separable in both types of consumption, both types of labor and in real money balances.*

This observation implies that we can divide our analysis of the dynamic system in two parts. First we analyze the system formed by (41) and (42) and then we proceed to analyze the system formed by (39) and (40).

The system of differential equations (41) and (42) can be linearized to obtain

$$\begin{pmatrix} \dot{\lambda} \\ \dot{b} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \psi\lambda^{ss} \\ \frac{1}{(\lambda^{ss})^2}\left(\alpha + \frac{(\theta_T\lambda^{ss})^{\frac{1}{1-\theta_T}}}{1-\theta_T}\right) & r \end{pmatrix}}_{F} \begin{pmatrix} \lambda - \lambda^{ss} \\ b - b^{ss} \end{pmatrix} \tag{43}$$

where the shadow price of wealth ($\lambda$) is a jump variable while the stock of foreign bonds ($b$) is a predetermined variable.

It is straightforward to prove that the determinant of the matrix $F$ of this linearized system is negative. That is $Det(F) = \upsilon_1 \upsilon_2 = -\frac{\psi}{\lambda^{ss}} \left( \alpha + \frac{(\theta_T \lambda^{ss})^{\frac{1}{1-\theta_T}}}{1-\theta_T} \right) < 0$, where $\upsilon_i$ denotes the *ith* eigenvalue of $F$. It is also simple to derive that both eigenvalues are real. Since the determinant is negative then there is one negative eigenvalue and one positive eigenvalue. Without loss of generality assume that $\upsilon_1 < 0$ and $\upsilon_2 > 0$. This result implies that the dynamics of this system can be characterized by a saddle-path whose equation can be described as

$$\lambda - \lambda^{ss} = -\frac{\psi \lambda^{ss}}{\upsilon_1}(b - b^{ss}) \tag{44}$$

This equation in tandem with the differential equation

$$\dot{b} = \upsilon_1 (b - b^{ss}) \tag{45}$$

and the initial condition $b_0$ allow us to reconstruct the dynamic paths of the shadow price of wealth ($\lambda$) and the stock of foreign bonds ($b$).

More importantly it can be seen that the interest rate feedback rule does not affect this system of two equations in (43). Therefore this analysis is valid regardless of the specification of the monetary rule followed by the government. Then it is clear that whether the interest rate rule affects the determinacy of equilibrium in this economy will completely depend on the system of equations (39) and (40).

This analysis also implies that we exclusively have to focus on the differential equations (39), (40) and (45); given that this last differential equation, equation (44) and the initial condition $b_0$ describe completely the system (43).

To continue our analysis we have to linearize the differential equations (39), (40), and (45). Equation (45) is already a linear differential equation. On the contrary, to linearize equations (39) and (40) we have to solve for the function $R = \hat{\rho}\,(b,\ \pi_N,\ \lambda,\ c_N,)$ using (36), (38) and (44). Doing so we obtain the linearized system

$$\begin{pmatrix} \dot{b} \\ \dot{\pi}_N \\ \dot{c}_N \end{pmatrix} = \underbrace{\begin{pmatrix} \upsilon_1 & 0 & 0 \\ 0 & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}}_{J} \begin{pmatrix} b - b^{ss} \\ \pi_N - \pi^{ss} \\ c_N - c_N^{ss} \end{pmatrix} \tag{46}$$

where

$$J_{22} = r > 0 \qquad J_{23} = \frac{\phi\,(c_N^{ss})^{\frac{1-\theta_N}{\theta_N}}}{\gamma \theta_N^2} < 0$$

$$J_{31} = \frac{\psi c_N^{ss}\left(\alpha\rho_\pi + \rho_\epsilon + \rho_{\dot{e}/e}\right)}{1-(\alpha\rho_\pi + \rho_\epsilon + \rho_{\dot{e}/e})}\left\{1 + \frac{\lambda^{ss}}{\nu_1}\left(\frac{\rho_y \alpha^2 \left(c_N^{ss}\right)^{1-\alpha}}{\left(\lambda^{ss}\right)^\alpha}\left(y_T^{ss} + \frac{c_N^{ss}}{e^{ss}}\right) - \frac{\rho_e c_N^{ss}}{1-\alpha}\right)\right\}$$

$$J_{32} = -\frac{c_N^{ss}\left(1-\rho_\pi - \rho_\epsilon - \rho_{\pi_N}\right)}{1-\alpha\rho_\pi - \rho_\epsilon - \rho_{\dot{e}/e}} \qquad J_{33} = \frac{\rho_y(1-\alpha)y^{ss} + \rho_e e^{ss}}{1-\alpha\rho_\pi - \rho_\epsilon - \rho_{\dot{e}/e}}$$

For simplicity we still keep the notation $\rho_x$ as the derivative of the Taylor Rule with respect to $x$, but in this case the derivative is evaluated at the steady state.

The matrix $J$ of system (46) is block triangular. Hence it is straightforward to see that one of the roots of the characteristic equation associated with the matrix is negative $\omega_1 = \upsilon_1 < 0$. Since the stock of foreign bonds ($b$) is a predetermined variable and since we know that there is always a negative root, $\omega_1 = \upsilon_1 < 0$, we can concentrate our determinacy analysis on the subsystem associated with the submatrix $J_s$,

$$\begin{pmatrix} \dot{\pi}_N \\ \dot{c}_N \end{pmatrix} = \underbrace{\begin{pmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{pmatrix}}_{J_s} \begin{pmatrix} \pi_N - \pi^{ss} \\ c_N - c_N^{ss} \end{pmatrix} \tag{47}$$

for which

$$Trace(J_s) = r + \frac{\rho_y(1-\alpha)y^{ss} + \rho_e e^{ss}}{1-\alpha\rho_\pi - \rho_\epsilon - \rho_{\dot{e}/e}} \tag{48}$$

$$Det(J_s) = r\frac{\rho_y(1-\alpha)y^{ss} + \rho_e e^{ss}}{1-\alpha\rho_\pi - \rho_\epsilon - \rho_{\dot{e}/e}} + \frac{\phi\left(c_N^{ss}\right)^{\frac{1}{\theta_N}}}{\gamma\theta_N^2}\frac{\left(1-\rho_\pi - \rho_\epsilon - \rho_{\pi_N}\right)}{(1-\alpha\rho_\pi - \rho_\epsilon - \rho_{\dot{e}/e})} \tag{49}$$

## 3.1 Simple Rules

In this section we present the most important results in terms of the interest-rate feedback rules that depend exclusively on one argument. The first propositions show that conditions under which these rules induce aggregate instability in the small open economy by generating multiple equilibria depend on the measure of inflation to which the Central Bank responds. We proceed to analyze an interest rate rule that depends solely on the non traded goods inflation rate, that is $\rho_{\pi_N} > 0$ and $\rho_\pi = \rho_\epsilon = \rho_e = \rho_{\dot{e}/e} = \rho_y = 0$.

**Proposition 1** *Assume that* $R = \rho(\pi_N)$ *with* $\rho_{\pi_N} > 0$,
***a)*** *If* $\rho_{\pi_N} < 1$ *(a passive rule in terms of the non-traded goods inflation rate) then there exists a continuum of perfect foresight equilibria in which* $\{\pi_N, c_N\}$ *converge asymptotically to the steady state.*
***b)*** *If* $\rho_{\pi_N} > 1$ *(an active rule in terms of the non-traded goods inflation rate) then there exists a unique perfect foresight equilibrium in which* $\{\pi_N, c_N\}$ *converge to the steady state.*

***Proof.*** *See Appendix* ■

Proposition 1 suggests that if the government wants to avoid multiple monetary equilibria, then its interest rate rule should be active in terms of the non-traded goods inflation rate. In other words, only rules that are passive with respect to the non-traded goods inflation rate may generate aggregate instability. The similarity of these results to the determinacy analysis results for rules in closed economies as in Clarida et al. (2000) is immediately obvious. But to grasp the intuition of these results we can derive and use the following two equations. The first one can be obtained applying the same procedure we used to derive equation (28) without imposing the simplifying assumption $\theta_N = 1$. Then we obtain

$$\pi_{Nt} = \int_t^{\infty} \mathsf{e}^{-r(s-t)} \frac{-\phi}{\gamma} \left( (c_{Ns})^{\frac{1}{\theta_N}} - (y_N^{ss})^{\frac{1}{\theta_N}} \right) ds \tag{50}$$

where $y_N^{ss} = \frac{(1+\phi)(1-\alpha)\theta_N}{\phi}$. Equation (50) implies that the inflation of non-traded goods at time $t$ is a positive function of the actual and the expected excesses of demand for non-traded goods.[25] The second equation is derived from the rule $R = \rho(\pi_N)$ together with (25) and it allows us to understand the dynamics of consumption of non-traded goods,

$$\frac{\dot{c}_N}{c_N} = (\rho(\pi_N) - r - \pi_N) \tag{51}$$

The resemblance of this equation to the equation for the evolution of consumption in the closed economy is interesting.[26] This resemblance is useful to define $\rho(\pi_N) - \pi_N$ as the real interest rate in terms of non-traded goods inflation. From (51) it is clear that the dynamics of this real interest rate will determine the dynamics of non-traded consumption.

For the purpose of showing the possibility of self-fulfilling equilibria under a passive rule let us assume that at time $t = 0$, all the agents in the economy expect a higher non-traded goods inflation, $\pi_{N0} > \pi_N^{ss} = 0$. If the rule is defined as $R = \rho(\pi_N)$, then the government responds increasing the nominal interest rate. But since the rule is passive ($\rho_{\pi_N} < 1$), then the real interest rate in terms of the non-traded goods inflation ($\rho(\pi_N) - \pi_N$) will actually decrease. By equation (51), this in turn implies that the growth rate of consumption of non-traded goods becomes negative $\left(\frac{\dot{c}_N}{c_N} < 0\right)$. However if consumption of non-traded goods decreases over time and converges to its steady-state level, it must be the case that at $t = 0$ this consumption jumps up. That is $c_{N0} > c_N^{ss} = y_N^{ss}$. This path of consumption means that the excess of demand for non-traded goods will be expected to be positive (see equation (50)). But if this is the case, at $t = 0$ firms will increase the price of non-traded goods since the demand for these goods is "too" high. The increase of prices of non-traded goods by the firms will validate the original expectations about a higher non-traded goods inflation.

[25] As was mentioned before this excess of demand is measured with respect to the steady state level $y_N^{ss}$.

[26] See Benhabib, Schmitt-Grohé and Uribe (2001a,b) for instance.

On the other hand, if the rule is active ($\rho_{\pi_N} > 1$) then the expectations of a higher non-traded goods inflation will be destroyed. In this case the real interest rate in terms of the non-traded goods inflation will increase and therefore consumption of non-traded goods will increase over time as well. Thus at $t = 0$, consumption of non-traded goods jumps down. That is $c_{N0} < c_N^{ss} = y_N^{ss}$, implying that the excess of demand for these goods is expected to be negative. Finally in response to this negative excess of demand, firms will decrease prices of non-traded goods. Thus the initial expectations of a higher non-traded goods inflation are not self-fulfilled.

We proceed to analyze rules in which the measure of inflation is the inflation of traded goods or in our setup, the nominal depreciation rate since there is a perfect exchange rate pass-through; that is $\rho_\epsilon > 0$ and $\rho_\pi = \rho_{\pi_N} = \rho_e = \rho_{\dot{e}/e} = \rho_y = 0$.

**Proposition 2** *If $R = \rho(\epsilon)$ then there exists a continuum of perfect foresight equilibria in which $\{\pi_N, c_N\}$ converge asymptotically to the steady state, regardless whether the policy is active ($\rho_\epsilon > 1$) or passive ($\rho_\epsilon < 1$) in terms of the nominal depreciation rate.*

***Proof.*** *See Appendix.* ■

The surprising result of Proposition 2 is that the rule under analysis always leads to multiple equilibria regardless of its response to the traded goods inflation rate. This result depends on the fact that there is perfect exchange rate pass-through and therefore the traded goods inflation rate coincides with the nominal depreciation rate. To explain the intuition of Proposition 2 it is useful to use equation (50) and to derive the following two equations. Since the rule is specified as $R = \rho(\epsilon)$ then we can rewrite (25) as

$$\frac{\dot{c}_N}{c_N} = (\rho(\epsilon) - r - \pi_N) \tag{52}$$

In addition we can use the revised interest parity condition (22) and $R = \rho(\epsilon)$ to obtain

$$r + \epsilon - \psi(b - b^{ss}) = \rho(\epsilon) \tag{53}$$

This revised UIP condition can be used to derive the dynamics of the nominal depreciation rate which in turn determines the dynamics of the nominal interest rate. More importantly neither the non-traded goods inflation nor the consumption of non-traded goods affect this condition. Therefore they do not influence the dynamics of the nominal depreciation rate and the nominal interest rate. Taking this into account we can construct the following self-fulfilling equilibrium. Assume that at time $t = 0$, all the agents of the economy expect a higher non-traded goods inflation rate, $\pi_{N0} > \pi_N^{ss} = 0$. Since the nominal interest rate is predetermined by the nominal depreciation rate obtained from (53), then the real interest rate in terms of the non-traded goods inflation $(\rho(\epsilon) - \pi_N)$ will go down. Therefore consumption of non-traded goods will decrease over time as can be deduced from (52). However if over time consumption of non-traded goods decreases converging to its steady-state level, then it must be the case that at $t = 0$, this consumption jumps up. That is $c_{N0} > c_N^{ss} = y_N^{ss}$. This consumption path implies that the excess of demand for non-traded goods will be expected to be positive. Therefore at $t = 0$ firms will respond increasing prices of non-traded goods validating the original expectations about a higher non-traded goods inflation rate (see equation (50)).

It is important to emphasize that Proposition 2 implies that the government should not target the inflation rate associated with traded goods, even if it follows an active rule. The reason is that this inflation rate is affected by the nominal depreciation rate. But it is precisely the direct response of the interest rate rule to the nominal depreciation rate what opens the possibility of multiple equilibria for active rules. This result is also interesting since it brings the attention upon the type of monetary policy followed by countries that have been hit by a currency crisis. In particular, Lahiri and Vegh (2000, 2003), and Flood and Jeanne (2001) have studied the fiscal costs and the output costs of defending the currency under attack through higher interest rates. Our proposition points out that there might be other costs. A government that exclusively follows a rule such that it increases the nominal interest rate whenever the nominal depreciation increases, can impose "the cost" of inducing instability in the economy by opening the possibility of self-fulfilling equilibria.[27]

We continue our determinacy of equilibrium analysis pointing out that conditions under which interest rate rules induce aggregate instability in the small open economy also depend on the degree of openness of the economy. To show this, we consider a simple rule whose sole argument corresponds to the CPI-inflation rate; that is $\rho_\pi > 0$ and $\rho_\epsilon = \rho_{\pi_N} = \rho_e = \rho_{\dot{e}/e} = \rho_y = 0$.

**Proposition 3** *Assume that* $R = \rho(\pi)$ *with* $\rho_\pi > 0$,
***a)*** *If* $\rho_\pi < 1$ *(a passive rule in terms of the CPI-inflation) then there exists a continuum of perfect foresight equilibria in which* $\{\pi_N, c_N\}$ *converge asymptotically to the steady state.*
***b)*** *If* $1 < \frac{1}{\alpha} < \rho_\pi$ *(an active rule in terms of the CPI-inflation) then there exists a continuum of perfect foresight equilibria in which* $\{\pi_N, c_N\}$ *converge asymptotically to the steady state.*
***c)*** *If* $1 < \rho_\pi < \frac{1}{\alpha}$ *(an active rule in terms of the CPI-inflation) then there exists a unique perfect foresight equilibrium in which* $\{\pi_N, c_N\}$ *converge to the steady state.*

***Proof.*** *See Appendix.* ■

Proposition 3 illustrates that the results from the determinacy analysis of interest rate rules for small open economies are not a simple extension of the ones for closed economies. How open

[27] Zanna (2003a) studies interest rate rules that only respond to the nominal depreciation rate. He shows that in a discrete time model forward-looking rules and contemporaneous rules always lead to real indeterminacy.

the economy is, that is how big the share of traded goods is, becomes a fundamental factor on the determinacy of equilibrium under *active* interest rate rules.

Figure 1 is a graphical representation of Proposition 3. This figure shows that in the small open economy active rules may lead to multiple equilibria (real indeterminacy) if the interest rate response coefficient to inflation is greater than the inverse of the share of traded goods. That is $\rho_\pi > \frac{1}{\alpha}$. To emphasize the importance of this result, assume that a small open economy follows the typical Taylor Rule whose interest rate response coefficient corresponds to $\rho_\pi = 1.5$ (see Taylor(1993)). The rule is clearly active with respect to the CPI-inflation. Then if the degree of openness of this economy is $\alpha \geq 0.67$, Proposition 3 implies that the rule may induce aggregate instability in the economy by generating multiple equilibria. The reason is that the rule may embark the economy on fluctuations that are not only determined by fundamentals but also by self-fulfilling expectations.

CPI-Inflation Coefficient in the Rule ($\rho_\pi$) vs. Share of Traded Goods ($\alpha$)

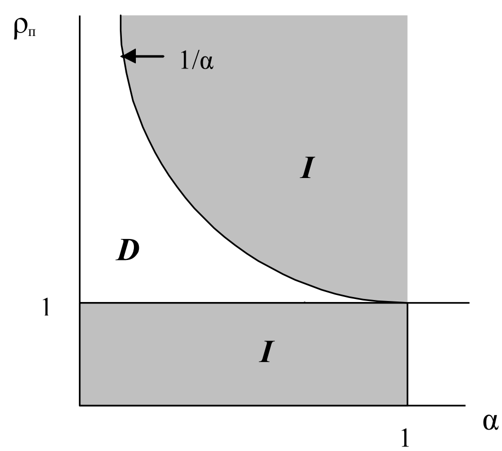


Figure 1: "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

To give a simple example of these expectation driven fluctuations we can follow Dupor (1999), and construct and simulate numerically a self-fulfilling equilibrium for an active interest rate rule. We choose Canada as the country and borrow the values of some of the parameters from different works. From Mendoza (1995) we borrow the labor income shares for both sectors, $\theta_T$ and $\theta_N$ and the value of the international real interest rate, $r$. From Schmitt-Grohé and Uribe (2001) we borrow the values of $b^{ss}$ and $\psi$. From Schmitt-Grohé (1997) we set $\phi$ such that the steady-state mark-up in the non-traded sector corresponds to 1.4. The steady-state inflation, $\pi^{ss}$, is calculated as the annual average inflation between 1980-2002. Moreover we set the price adjustment cost parameter

$\gamma$, such that the average time of changing a price in the non-traded sector is one year.[28] Following Devereux (2001) we set the share of traded goods to $\alpha = 0.5$ and using Proposition 3, we set $\rho_\pi$ to be greater than $\frac{1}{\alpha}$. In this example we use $\rho_\pi = 2.1$ and assume that at time zero people expect a higher non-traded goods inflation, that is $\pi_{N0} = 4.6\% > 4.1\% = \pi^{ss}$ in order to construct a self-fulfilling equilibrium. Table 1 presents the numerical values assigned to the parameters.

**Table 1: Parametrization**

| $\theta_N$ | $\theta_T$ | $\rho_\pi$ | $r$ | $\phi$ | $\gamma$ | $\alpha$ | $\psi$ | $R^{ss}$ | $\pi^{ss}$ | $b^{ss}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.56 | 0.51 | 2.2 | 0.04 | $-3.5$ | 27 | 0.5 | 0.00074 | 0.081 | 0.041 | 0.7442 |

The dynamic responses of the real interest rate (in terms of the non-traded goods inflation rate), the non-traded consumption and the traded goods inflation rate to these higher expectations are presented in Figure 2. The responses are measured as deviations from the steady state. What is important to emphasize in this simple example is that these fluctuations are mainly generated and driven by self-fulfilling expectations. It is in this sense that rules that lead to multiple equilibria may destabilize the economy. Under higher expectations of non-traded goods inflation, the government reduces the real interest rate in terms of the non-traded goods ($R - \pi_N$). But a reduction of this real interest rate leads to a negative growth rate of consumption of non-traded goods. Hence if consumption of non-traded goods decreases over time and converges asymptotically to its steady state level, this consumption must jump up at time zero. The increase in this consumption at time zero and its evolution over time lead to a positive expected excess of demand of non-traded goods. In response to this excess of demand firms increase the price of non-traded goods and end validating the original people's expectations about a higher non-traded goods inflation.

Although the preceding analysis shows that it is possible to construct a self-fulfilling equilibrium when the interest rate rule is active and such that $\rho_\pi > \frac{1}{\alpha}$, it may be unclear why openness ($\alpha$) matters for the equilibrium analysis. In order to answer this question it is important to recall that the CPI-inflation rate is a weighted average of the nominal depreciation rate and the non-traded goods inflation rate; where the weight of the nominal depreciation rate corresponds to $\alpha$, the degree of openness, while the weight of the non-traded goods inflation rate corresponds to $1-\alpha$. Then the possibility of real indeterminacy under active rules with respect to the CPI-inflation stems from the direct effect that the nominal depreciation has on the CPI-inflation. The more open the economy is (that is the greater $\alpha$ is), the greater this direct effect is and therefore the higher the possibility of having multiple equilibria under active rules. In the extreme case when the degree of openness

[28] Dib's (2001) estimates of $\gamma$ for Canada vary between 2.80 and 44.07, depending on the model specification (type of nominal and real rigidities).

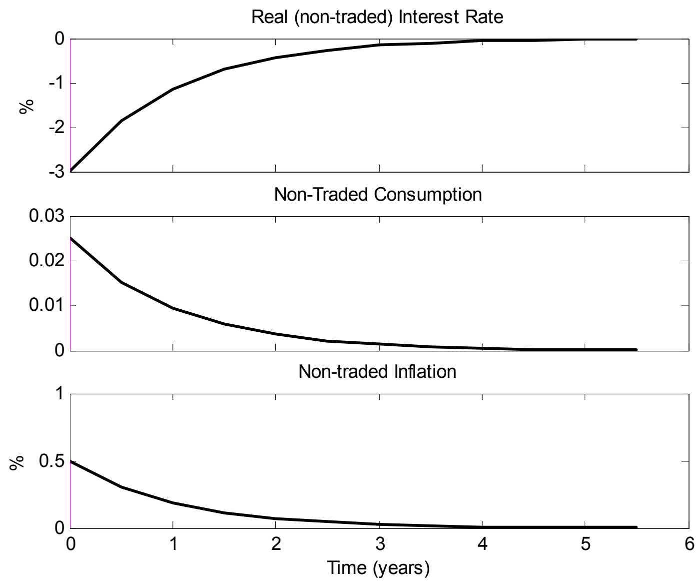


Figure 2: Impulse responses when at $t = 0$ people expect a higher non-traded goods inflation $\pi_{N0} = 4.6\% > 4.1\% = \pi^{ss}$

$\alpha$ is close to 1, and therefore the CPI-inflation rate coincides with the nominal depreciation rate, Proposition 2 states that multiple equilibria arise under active rules (see Figure 1). On the other hand, in the extreme case when the degree of openness $\alpha$ is close to 0, and therefore the CPI-inflation rate coincides with the non-traded goods inflation rate, Proposition 1 establishes that a unique equilibrium arises under active rules (see Figure 1).

Proposition 3 has two important consequences. First it suggests to revise the interpretation of some of the estimations for interest rate rules for small open economies. In particular empirical works like Clarida et al. (1998) have claimed that active rules ($\rho_\pi > 1$) are important since they induce stability in inflation and in the whole economy. Proposition 3 shows that this claim is not necessarily true. Second, if one is interested in drawing any conclusion about real indeterminacy induced by the rule our results support the line of research started by Lubik and Schorfheide (2003a) for closed economies. That is, our result points out the necessity of having an estimate not only of the parameters of the interest rate rules but also of the structural parameters of the model such as the share of traded goods. In this sense the result questions the univariate monetary policy estimations for open economies.

Table 2 summarizes the results of our analysis of simple rules in Proposition 1, 2 and 3. It shows how conditions under which interest rate rules lead to real indeterminacy depend not only on the type of monetary rule, active or passive, but also on how open the economy is and on the measure of inflation to which the central bank responds.

**Table 2: Simple Rules**

$R = \rho(x), \ \ with \ \ \rho_x > 0$

| | Measure of Inflation $x$ | | |
|---|---|---|---|
| | $CPI$ | $Non-Traded$ | $Traded$ |
| $Monetary\ Policy$ | $\pi$ | $\pi_N$ | $\epsilon$ |
| $Passive\ (\rho_x < 1)$ | $I$ | $I$ | $I$ |
| $Active\ (\rho_x > 1)$ | $I\ or\ D$ | $D$ | $I$ |
| $1 < \rho_x < \frac{1}{\alpha}$ | $D$ | $D$ | $I$ |
| $\frac{1}{\alpha} < \rho_x$ | $I$ | $D$ | $I$ |

$Note: \epsilon = \pi_T;\ D\ stands\ for\ real\ determinacy; I,\ for\ real\ indeterminacy;\ and\ \alpha\ for\ the\ degree\ of\ openness$

A simple inspection of the results in Table 2 suggests that in order to avoid multiple equilibria, governments in small open economies should design rules satisfying two requirements. First the measure of inflation of the rule should be the non-traded goods inflation or at least a measure of inflation that is not heavily affected by the nominal depreciation rate. Second, the rule should be active. Surprisingly this result coincides with some of the empirical and theoretical results of the aforementioned literature. In particular Clarida, Gali and Gertler (2001) and Kollmann (2002) emphasize the fact that openness raises the important distinction between the domestic inflation (in other contexts the non-traded goods inflation) and the CPI-inflation that is affected by changes of the nominal exchange rate. In their models, to the extent that there is a perfect exchange rate pass-through, the government should target the domestic inflation making it the measure of inflation that should be taken into account in the design of the rule.[29] However it is important to notice that these works arrive at these conclusions *without* pursuing an equilibrium determinacy analysis as we do in the present paper.

[29] A similar proposal by Ball (1999) points out the importance of targeting a modified inflation index that filters out the transitory effects of exchange rate movements, or to use an average of CPI-inflation over a longer period.

### 3.2 Extended Rules

In this part of the paper we study interest-rate feedback rules that include more than one argument. Besides a measure of inflation (CPI-inflation rate, $\pi$, or non-traded goods inflation rate, $\pi_N$), the rule may include the output ($y$), the nominal depreciation rate ($\epsilon$), the real exchange rate ($e$) or the real depreciation rate ($\dot{e}/e$).

The first important result is that openness ($\alpha$) is still a determinant factor in the equilibrium analysis of active rules. In particular any extended rule that besides the CPI-inflation includes any of the aforementioned variables will lead to multiple equilibria if the rule is active with respect to the CPI-inflation rate and such that $\rho_\pi > \frac{1}{\alpha}$.

**Proposition 4** *If $R = \rho(\pi, \epsilon, e, \frac{\dot{e}}{e}, y)$ and $\rho_\pi > 0$, $\rho_\epsilon > 0$, $\rho_e > 0$, $\rho_{\dot{e}/e} > 0$, $\rho_y > 0$ and $\rho_\pi > \frac{1}{\alpha}$ then there exists a continuum of perfect foresight equilibria in which $\{\pi_N, c_N\}$ converge asymptotically to the steady state.*

***Proof.*** *See Appendix.* ■

Proposition 4 is important because it calls for a revision of some of the proposals from previous literature about extended interest rate rules in the small open economy. Clarida et al (1998), Ball (1999), Svensson (2000), Taylor (1999b), Monacelli (1999) and Kollmann (2002) have studied rules that not only include a measure of inflation and the output gap, but also include the real exchange rate or the real depreciation rate. For instance Svensson (2000) suggests that flexible CPI-inflation targeting and its derived optimal interest rate rule that includes the real exchange rate, stands out in limiting the variability not only of the CPI inflation but also of the output gap and the real exchange rate. In the same line, Monacelli (1999) proposes a rule that includes the nominal depreciation rate in order to reduce the volatility of the CPI-inflation and the domestic inflation. Proposition 4 brings the attention upon these proposals since it is possible that an active interest rate rule with respect to the CPI-inflation rate may lead to multiple equilibria, regardless of how the rule responds to the nominal depreciation rate, the real exchange rate and/or the real depreciation rate. If this is the case, it is feasible to construct an equilibrium that increases the volatilities of the CPI-inflation rate, the domestic inflation rate, the output gap and the real exchange rate.

In Zanna (2003b) we pursue the study of different specifications of the general rule presented in Proposition 4. As expected from the study of simple rules, the determinacy of equilibrium analysis not only depends on the response coefficients to the arguments of the rule, but also on the degree of openness of the economy and on the measure of inflation used in the rule. Among these different specifications it is probably worth presenting the case of a rule in which the interest rate responds to both a measure of inflation and the nominal depreciation rate. To motivate the study of these rules we recall that Calvo and Reinhart (2002) have pointed out that, surprisingly, emerging economies that claim to allow their exchange rate to float, mostly do not. They suffer of what they call "fear of floating" since governments may be concerned about inflation and, in particular, about the effect of changes of the nominal exchange rate on the CPI-inflation rate. Calvo and Reinhart actually find that the relative high variability of nominal and real interest rates of the "feared" economies suggests that they are not relying exclusively on intervening the foreign exchange rate market to smooth the path of the exchange rate. On the contrary, they observe that the nominal interest rate has become a common instrument to smooth the fluctuation of the exchange rate.

In empirical terms, Ades, Buscaglia and Masih (2002) and Zanna (2003b) have estimated interest rate reaction functions that include the nominal depreciation rate or a deviation of the nominal exchange rate from its long-run level for some emerging economies. They find evidence of "fear of floating". On the other hand, Lubik and Schorfheide (2003) have tested if Central Banks in Canada, New Zealand, Australia and UK are following interest rate rules that target the nominal exchange rate. They also find evidence that supports the importance of studying the determinacy of equilibrium under rules that respond to both the CPI-inflation rate and the nominal depreciation rate. The following proposition accomplishes this goal.

**Proposition 5** *Assume that* $R = \rho(\pi, \epsilon)$ *with* $\rho_\pi > 0$ *and* $\rho_\epsilon > 0$,
***a)*** *If either* $\rho_\pi + \rho_\epsilon < 1$ *or* $1 < \alpha\rho_\pi + \rho_\epsilon$ *then there exists a continuum of perfect foresight equilibria in which* $\{\pi_N, c_N\}$ *converge asymptotically to the steady state.*
***b)*** *If* $\alpha\rho_\pi + \rho_\epsilon < 1 < \rho_\pi + \rho_\epsilon$ *then there exists a unique perfect foresight equilibrium in which* $\{\pi_N, c_N\}$ *converge to the steady state.*

***Proof.*** *See Appendix.* ■

Figure 3 is a graphical representation of the results in Proposition 5. It can be observed that rules that are passive with respect to both the CPI-inflation rate and the nominal depreciation rate and such that $\rho_\pi + \rho_\epsilon < 1$, may open the possibility of multiple equilibria in the economy. However policies that are active with respect to the same both arguments and such that $1 < \alpha\rho_\pi + \rho_\epsilon$ may also cause multiple equilibria. To see the importance of this result assume an economy whose share of traded goods is close to $\alpha = 0.5$ and suppose that the policy makers in this economy follow a typical Taylor rule with $\rho_\pi = 1.5$. In order to induce aggregate instability in this economy by generating multiple equilibria, it is sufficient that the government increases the nominal interest rate by more than 0.33% ($\rho_\epsilon > 0.33$) in response to a 1% increase in the nominal depreciation rate.

Once more, the possibility of real indeterminacy for active rules is due to the presence of the nominal depreciation rate in the rule. But in this case this effect is direct since $\rho_\epsilon \neq 0$. In particular if the rule is active with respect to the nominal depreciation rate ($\rho_\epsilon > 1$), multiple equilibria arise, regardless of how active or passive the rule is with respect to the CPI-inflation.

In addition it is important to observe that if the share of the traded goods is close to one, any interest rate rule defined as $R = \rho(\pi, \epsilon)$ will cause real indeterminacy no matter how responsive it

CPI-Inflation Coefficient ($\rho_\pi$) vs. Nominal Depreciation Cofficient ($\rho_\varepsilon$)

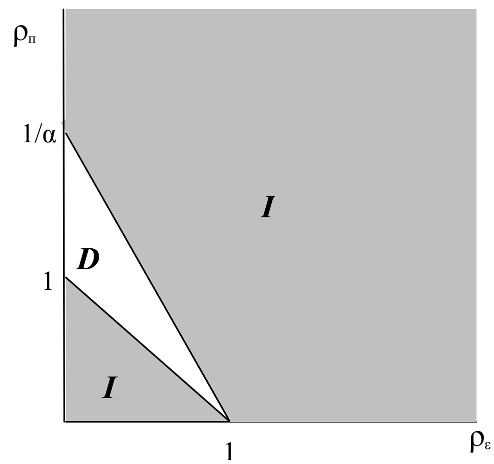


Figure 3: "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

is to both arguments. Hence Proposition 5 implies that the "fear of floating" can be pervasive in small and very open economies, since it is likely that a government whose rule reacts to the nominal depreciation will destabilize the economy. However a rule that depends on the CPI-inflation and on the nominal depreciation rate is implicitly assuming that such a government is a myopic one. The reason is that the CPI-inflation is already affected by nominal depreciation rate. Therefore it may be important to analyze interest-rate feedback rules that depends on the nominal depreciation rate, but whose measure of inflation is the non-traded goods inflation rate instead of the CPI-inflation rate. In terms of the model we have that $\rho_{\pi_N} > 0$ and $\rho_\epsilon > 0$ and $\rho_\pi = \rho_e = \rho_{\dot{e}/e} = \rho_y = 0$.

**Proposition 6** *Assume that* $R = \rho(\pi_N, \epsilon)$ *with* $\rho_{\pi_N} > 0$ *and* $\rho_\epsilon > 0$,
***a)*** *If either* $\rho_{\pi_N} + \rho_\epsilon < 1$ *or* $\rho_\epsilon > 1$ *then there exists a continuum of perfect foresight equilibria in which* $\{\pi_N, c_N\}$ *converge asymptotically to the steady state.*
***b)*** *If* $1 < \rho_{\pi_N} + \rho_\epsilon$ *and* $\rho_\epsilon < 1$ *then there exists a unique perfect foresight equilibrium in which* $\{\pi_N, c_N\}$ *converge to the steady state.*

***Proof.*** *See Appendix.* ■

Figure 4 is a graphical representation of the results in Proposition 6. The first important observation about Proposition 6 is that the determinacy analysis does not depend on the degree of openness of the economy ($\alpha$). Second the proposition has an important message for policy makers. It conveys the idea that interest rate rules that are active with respect to the nominal depreciation

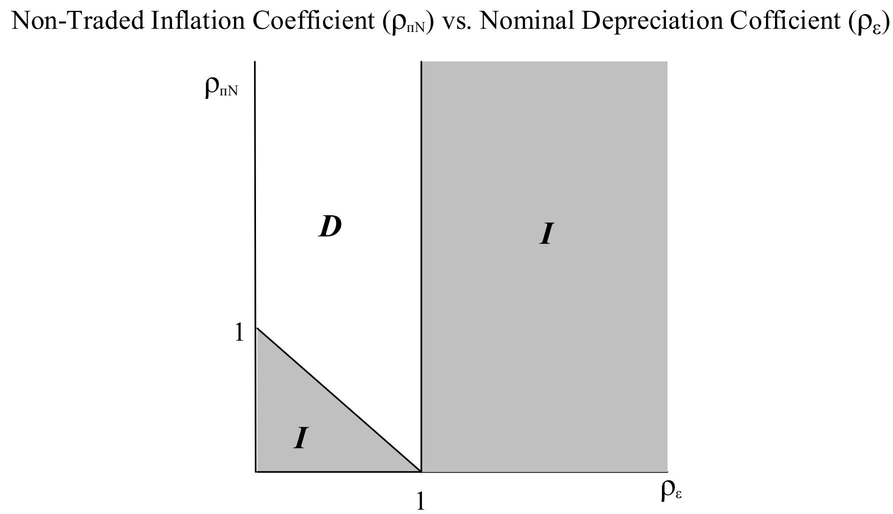


Figure 4: "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

rate ($\rho_\epsilon > 1$) lead to multiple monetary equilibria regardless of how responsive the interest rate is to the non-traded inflation. In the same line policies that are passive with respect to the non-traded inflation rate and to the nominal depreciation rate and such that $\rho_{\pi_N} + \rho_\epsilon < 1$ may also lead to real indeterminacy. These results suggest that if a government practices "dirty floating" or suffers from "fear of floating" in order to avoid destabilize to the economy, the policy maker should design an interest rate rule that is passive with respect to the nominal depreciation rate ($\rho_\epsilon < 1$) and active with respect to the non-traded inflation rate ($\rho_{\pi_N} > 1$).

### 3.3 Backward-Looking and Forward-Looking Rules

In this part of the paper we study interest-rate feedback rules that may include backward-looking or forward-looking elements. We will still assume that the monetary policy takes the form of an interest-rate feedback rule whereby the nominal interest rate is set as a function of one or more variables. But in this case, the possible variables are: the contemporaneous CPI-inflation rate ($\pi$), the weighted average of past interest rates ($R_p$), the weighted average of expected future rates of the CPI-inflation ($\pi_f$) and the weighted average of past rates of the CPI-inflation ($\pi_p$). The general form of the rule can be written as

$$R = \rho(\pi, R_p, \pi_f, \pi_p)$$

where $\rho(.)$ is continuous and strictly positive in all its arguments. It is also assumed to be non-decreasing in $\pi$, $R_p$, $\pi_f$ and $\pi_p$.

To understand the new arguments of the interest-rate feedback rule we provide some definitions. We define the weighted average of past interest rates, $R_p$, as

$$R_p = k_R \int_{-\infty}^{t} R(s)\mathrm{e}^{k_R(s-t)}ds \qquad k_R > 0 \tag{54}$$

where the parameter $k_R$ measures the weight that the monetary authority puts on interest rates observed in the past. If $k_R$ is large then the central bank puts a large weight on interest rates observed in the recent past.

In addition, the weighted average of expected future rates of the CPI-inflation, $\pi_f$, is defined as

$$\pi_f = k_f \int_{t}^{\infty} \pi(s)\mathrm{e}^{-k_f(s-t)}ds \qquad k_f > 0 \tag{55}$$

where the parameter $k_f$ measures the weight that the monetary authority puts on inflation rates of the future. If $k_f$ is small then the central bank puts a large weight on inflation rates of the distant future.

On the other hand, we define the weighted average of past rates of the CPI inflation, $\pi_p$, as

$$\pi_p = k_p \int_{-\infty}^{t} \pi(s)\mathrm{e}^{k_p(s-t)}ds \qquad k_p > 0 \tag{56}$$

where the parameter $k_p$ measures the weight that the monetary authority puts on inflation rates observed in the past. If $k_p$ is large then the central bank puts a large weight on inflation rates observed in the recent past.

Taking the derivative with respect to time to both sides of (54), (55) and (56), and applying Leibniz's rule we can find differential equations that will be useful to describe the dynamics of $R_p$, $\pi_f$ and $\pi_p$ respectively,

$$\dot{R} = k_R(R - R_p) \tag{57}$$

$$\dot{\pi}_f = k_f(\pi_f - \pi) \tag{58}$$

$$\dot{\pi}_p = k_p(\pi - \pi_p) \tag{59}$$

We can proceed to do the determinacy of equilibrium analysis. First, we analyze the particular interest-rate feedback rule that responds to both the CPI-inflation rate ($\pi$) and the weighted average of past interest rates ($R_p$). Then we study a pure forward-looking interest rate rule. That is a rule whose sole argument corresponds to the weighted average of expected future rates of the

CPI-inflation ($\pi_f$). Lastly, we analyze a pure backward-looking rule whose sole argument is the weighted average of past rates of the CPI-inflation ($\pi_p$).

To pursue the determinacy of equilibrium analysis for these rules it is important to observe that it is still valid to separate the analysis in two *steps: 1 and 2.* In other words we can proceed as we did in the study of simple and extended interest rate rules. The *steps* are the following:

*Step1*: we study the dynamics for the shadow price of wealth ($\lambda$) and the stock of foreign bonds ($b$). The reason is that the differential equations of these two variables are still independent of variables that include backward-looking and forward-looking elements such as $R_p$, $\pi_f$, and $\pi_p$.

*Step 2:* we focus on the dynamics of the non-traded goods consumption ($c_N$), the non-traded goods inflation rate ($\pi_N$) and a third variable like the weighted average of past interest rates of ($R_p$), or the weighted average of expected future CPI-inflation rates ($\pi_f$) or the weighted average of past CPI-inflation rates ($\pi_p$).

As was mentioned above, we start analyzing the rule that besides the CPI-inflation rate, includes the weighted average of past interest rates of ($R_p$), that is

$$R = \rho(\pi, R_p) \tag{60}$$

The motivation for this type of rules comes from Goodfriend (1991) and English et al. (2002) who have observed the central banks tendency of smoothing interest rates. Moreover on theoretical grounds, Rotemberg and Woodford (1999) and Giannoni and Woodford (2002) have argued that the performance of Taylor Rules can be improved by adding lagged values of the nominal interest rate. In particular Giannoni and Woodford have suggested that the coefficient on the lagged interest rate should be greater than one.

In *Step 1* we obtain the same results as we did before, analyzing the system (43). For this system there is one jump variable, $\lambda$, and one predetermined variable, $b$, and there is one positive real eigenvalue and one negative real eigenvalue. Moreover the steady state is described as a saddle path that is independent of the monetary rule followed by the government. This result implies that the determinacy of equilibrium properties will depend exclusively on the dynamic subsystem related to the variables $c_N$, $\pi_N$ and $R_p$.

In *Step 2* we pursue a determinacy of equilibrium analysis using the differential equations (25), (26), (45), and (57) in tandem with (36) and (60). Linearizing these differential equations and using linearized versions of (36) and (60) allow us to obtain the following system

$$\begin{pmatrix} \dot{b} \\ \dot{R}_p \\ \dot{\pi}_N \\ \dot{c}_N \end{pmatrix} = \underbrace{\begin{pmatrix} \upsilon_1 & 0 & 0 & 0 \\ N_{21} & N_{22} & N_{23} & 0 \\ 0 & 0 & N_{33} & N_{34} \\ N_{41} & N_{42} & N_{43} & 0 \end{pmatrix}}_{N} \begin{pmatrix} b - b^{ss} \\ R_p - R^{ss} \\ \pi_N - \pi^{ss} \\ c_N - c_N^{ss} \end{pmatrix} \tag{61}$$

where

$$N_{21} = \frac{k_R \rho_\pi \psi}{(\frac{1}{\alpha} - \rho_\pi)} \qquad N_{22} = \frac{k_R(\rho_R + \alpha\rho_\pi - 1)}{(1 - \alpha\rho_\pi)} \qquad N_{23} = \frac{(1-\alpha)\rho_\pi k_R}{(1 - \alpha\rho_\pi)}$$

$$N_{33} = r > 0 \qquad N_{34} = \frac{\phi\,(c_N^{ss})^{\frac{1-\theta_N}{\theta_N}}}{\gamma\theta_N^2} < 0$$

$$N_{41} = \frac{c_N^{ss}\rho_\pi\psi}{(\frac{1}{\alpha} - \rho_\pi)} \qquad N_{42} = \frac{c_N^{ss}\rho_R}{1 - \alpha\rho_\pi} \qquad N_{43} = -\frac{c_N^{ss}(1 - \rho_\pi)}{1 - \alpha\rho_\pi}$$

It is straightforward to see that one of the roots of the characteristic equation associated with the matrix $N$ of system (61) is negative, $\omega_1 = \upsilon_1 < 0$. This is due to the fact that the matrix $N$ is block triangular. Since the stock of foreign bonds ($b$) is a predetermined variable and since we know that there is always a negative root, $\omega_1 = \upsilon_1 < 0$, we can focus our determinacy analysis on the subsystem

$$\begin{pmatrix} \dot{R}_p \\ \dot{\pi}_N \\ \dot{c}_N \end{pmatrix} = \underbrace{\begin{pmatrix} N_{22} & N_{23} & 0 \\ 0 & N_{33} & N_{34} \\ N_{42} & N_{43} & 0 \end{pmatrix}}_{N_s} \begin{pmatrix} R_p - R^{ss} \\ \pi_N - \pi^{ss} \\ c_N - c_N^{ss} \end{pmatrix} \tag{62}$$

The following proposition summarizes the determinacy analysis for the rule that, besides the CPI-inflation, includes past interest rates.

**Proposition 7** *Assume that $R = \rho(\pi, R_p)$ with $\rho_\pi > 0$ and $\rho_R > 0$,*
***a)*** *If either $\rho_\pi > \frac{1}{\alpha}$ (an active rule in terms of the CPI-inflation) or $\rho_\pi + \rho_R < 1$ then there exists a continuum of perfect foresight equilibria in which $\{R_p, \pi_N, c_N\}$ converge asymptotically to the steady state.*
***b)*** *If either $1 < \rho_\pi < \frac{1}{\alpha}$ (an active rule in terms of the CPI-inflation) or $1 - \rho_R < \rho_\pi < 1$ (a passive rule in terms of the CPI-inflation) then there exists a unique perfect foresight equilibrium in which $\{R_p, \pi_N, c_N\}$ converge asymptotically to the steady state.*

***Proof.*** *See Appendix.* ■

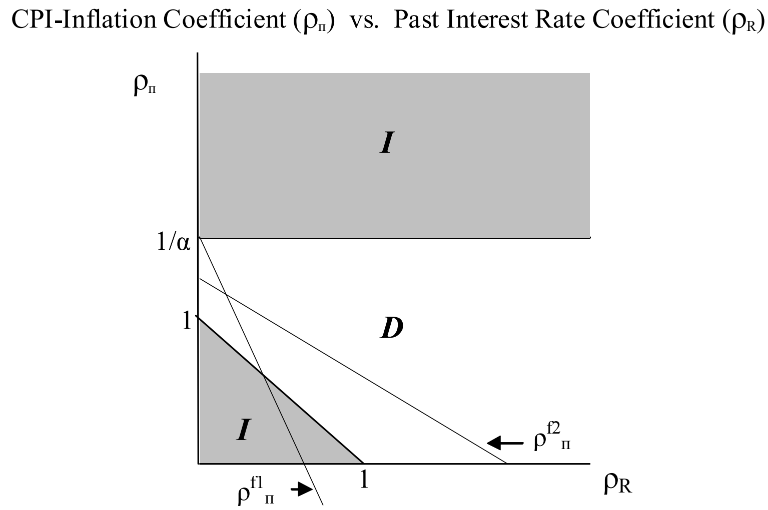


Figure 5: "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

Figure 5 is a graphical representation of the results in Proposition 7. This proposition suggests that even if the rule includes the weighted average of past interest rates, openness ($\alpha$) continues to be a fundamental determinant of the equilibrium under active interest rate rules with respect to the CPI-inflation. In particular if $\rho_\pi > \frac{1}{\alpha}$ then multiple equilibria are possible regardless of the response of the interest rate to past interest rates. This result is very important since it contrasts with the results in the closed economy literature. In particular Rotemberg and Woodford (1999) and Giannoni and Woodford (2002) have shown that rules with smoothing coefficient that is greater than one guarantee a locally unique equilibrium and are, in addition, capable of implementing the optimal real allocation. Proposition 5 points out that interest rate smoothing may be important for the determinacy of equilibrium. But it says that in the small open a smoothing coefficient that is greater than one is not a sufficient condition but a necessary condition to obtain a unique equilibrium.

We proceed to analyze rules that depend on the weighted average of the expected future CPI-inflation rates such as $R = \rho(\pi_f)$. The motivation of this analysis can arise from empirical works like Orphanides (1997) and Clarida et al. (2000). They argue that the central bank behavior in industrialized economies is primarily forward-looking.

The determinacy analysis of these rules can be pursued following the same steps as we did to study interest rate rules that depended on the weighted average of past interest rates but using (58) instead of (57). In this case we obtain the following linearized system,

$$\begin{pmatrix} \dot{b} \\ \dot{\pi}_f \\ \dot{\pi}_N \\ \dot{c}_N \end{pmatrix} = \underbrace{\begin{pmatrix} \upsilon_1 & 0 & 0 & 0 \\ \alpha k_f \psi & k_f(1-\alpha\rho_{\pi_f}) & -k_f(1-\alpha) & 0 \\ 0 & 0 & r & \frac{\delta}{c_N^{ss}} \\ 0 & c_N^{ss}\rho_{\pi_f} & -c_N^{ss} & 0 \end{pmatrix}}_{L} \begin{pmatrix} b - b^{ss} \\ \pi_f - \pi^{ss} \\ \pi_N - \pi^{ss} \\ c_N - c_N^{ss} \end{pmatrix} \tag{63}$$

where $\delta = \frac{\phi\left(c_N^{ss}\right)^{\frac{1}{\theta_N}}}{\gamma\theta_N^2} = \frac{(1+\phi)(1-\alpha)}{\gamma\theta_N} < 0$

As in (61), it is straightforward to see that one of the roots of the characteristic equation associated with the matrix of system (63) is negative, $\omega_1 = \upsilon_1 < 0$. Therefore given that $L$ is block triangular, that the stock of foreign bonds ($b$) is a predetermined variable and that there is always a negative root ( $\omega_1 = \upsilon_1 < 0$), we can focus our determinacy analysis on the subsystem

$$\begin{pmatrix} \dot{\pi}_f \\ \dot{\pi}_N \\ \dot{c}_N \end{pmatrix} = \underbrace{\begin{pmatrix} k_f(1-\alpha\rho_{\pi_f}) & -k_f(1-\alpha) & 0 \\ 0 & r & \frac{\delta}{c_N^{ss}} \\ c_N^{ss}\rho_{\pi_f} & -c_N^{ss} & 0 \end{pmatrix}}_{L_s} \begin{pmatrix} \pi_f - \pi^{ss} \\ \pi_N - \pi^{ss} \\ c_N - c_N^{ss} \end{pmatrix} \tag{64}$$

where $\delta = \frac{\phi\left(c_N^{ss}\right)^{\frac{1}{\theta_N}}}{\gamma\theta_N^2} = \frac{(1+\phi)(1-\alpha)}{\gamma\theta_N} < 0$. The following two propositions summarize the determinacy analysis for a pure forward-looking rule.

**Proposition 8** *Define $\Omega$ and $\rho^L$ as $\Omega = \frac{-(1+\phi)}{\theta_N\gamma r k_f}$ and*

$$\rho^L = \frac{1}{\alpha} + \frac{\Omega(1-\alpha)^2}{2\alpha^2} + \frac{r}{2\alpha k_f} - \frac{\sqrt{(\Omega(1-\alpha)^2 k_f + \alpha r)^2 + 4\Omega k_f \alpha(1-\alpha)((1-\alpha)k_f + \alpha r)}}{2\alpha^2 k_f}$$

*Assume that $R = \rho(\pi_f)$ and $\rho_{\pi_f} > 0$ and $0 < \Omega < 1$*

***a)*** *If $\rho_{\pi_f} < 1$ (a passive rule in terms of the weighted average of expected future CPI-inflation rates) then there exists a continuum of perfect foresight equilibria in which $\{\pi_f, \pi_N, c_N\}$ converge asymptotically to the steady state.*

***b)*** *If $1 < \rho_{\pi_f} < \rho^L$ (an active rule in terms of the weighted average of expected future CPI-inflation rates) then there exists a unique perfect foresight equilibrium in which $\{\pi_f, \pi_N, c_N\}$ converge to the steady state.*

***c)*** *If $\rho^L < \rho_{\pi_f}$ (an active rule in terms of the weighted average of expected future CPI-inflation rates) then there exists a continuum of perfect foresight equilibria in which $\{\pi_f, \pi_N, c_N\}$ converge asymptotically to the steady state*[30]*.*

***Proof.*** *See Appendix.* ■

[30] This is an active rule because $\rho^L \geq 1$ as we will show in the proof.

Pure Forward-Looking Rules
Inflation Coefficient ($\rho_{\pi f}$) vs. Share of Traded Goods ($\alpha$)

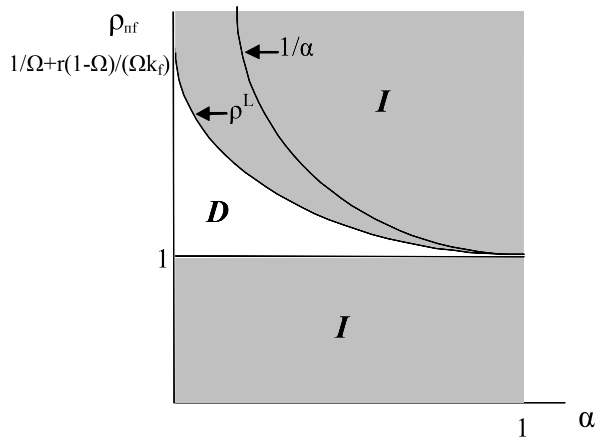


Figure 6: "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

Figure 6 is a graphical representation of the results in Proposition 8. It emphasizes that even in the case of a pure forward looking rule, openness ($\alpha$) is still a fundamental determinant of the equilibrium under active interest rate rules. In particular it is still valid that if $\rho_{\pi_f} > \frac{1}{\alpha}$ then the rule leads to real indeterminacy. However there is an important difference with respect to the results in Proposition 3. In particular, there are pure and active forward-looking rules such that $1 < \rho^L < \rho_{\pi_f} < \frac{1}{\alpha}$ that lead to multiple equilibria (see Figure 6); whereas in Proposition 3, any rule such that $1 < \rho_\pi < \frac{1}{\alpha}$ guaranteed a unique equilibrium. This main difference is due to the pure forward-looking character of the rule. In other words the parameter $k_f$ becomes also an important determinant of the equilibrium. If the central bank puts a large weight on inflation rates observed in the recent future and therefore $k_f$ is large, then the pure forward looking rule $R = \rho(\pi_f)$ will become similar to a simple rule $R = \rho(\pi)$.

What is important in the equilibrium analysis is not the absolute magnitude of $k_f$ but, instead, its relative value with respect to $\frac{-(1+\phi)}{\theta_N \gamma r}$. In other words when we say that the central bank puts a large weight on inflation rates observed in the recent future, we actually mean that $k_f$ is large enough to satisfy $\Omega = \frac{-(1+\phi)}{\theta_N \gamma r k_f} < 1$. On the other hand if $k_f$ is very small then the central bank puts a large weight on inflation rates of the distant future, where this big weight is determined by $\Omega = \frac{-(1+\phi)}{\theta_N \gamma r k_f}$. This means that the rule will be excessively forward-looking if $k_f$ is small enough such that $\Omega = \frac{-(1+\phi)}{\theta_N \gamma r k_f} > 1$. To see the importance of this observation we present the following proposition.

**Proposition 9** *Define* $\Omega$ *as* $\Omega = \frac{-(1+\phi)}{\theta_N \gamma r k_f}$. *If* $R = \rho(\pi_f)$ *and* $\rho_{\pi_f} > 0$ *and* $\Omega > 1$ *then there exists a continuum of perfect foresight equilibria in which* $\{\pi_f, \pi_N, c_N\}$ *converge asymptotically to the steady state.*

***Proof.*** *See Appendix.* ■

This proposition pursues an equilibrium analysis for those forward-looking rules in which the monetary authority puts a big weight on future expected CPI-inflation rates. It points out that excessively forward-looking rules will lead to multiple equilibria regardless of how active or passive the rule is. To have an idea of the real implications of this proposition we can use the aforementioned parametrization with the fact that the average forecast length of inflation is given by $k_f \int_0^\infty s\mathrm{e}^{-k_f s} ds = 1/k_f$ years. Then applying the results of this proposition we know that the rule always leads to multiple equilibria if the average forecast length of inflation to which the government responds is greater than 0.24 years $\left(\frac{1}{k_f} > \frac{\theta_N \gamma r}{-(1+\phi)} > 0.24\right)$.

The last interest rate feedback rule that we are interested in studying corresponds to a pure backward-looking rule that responds only to the weighted average of past CPI-inflation rates, $R = \rho(\pi_p)$. The motivation of this analysis can arise from the seminal paper by Taylor (1993).

Using (59) and following the same steps as before we obtain

$$\begin{pmatrix} \dot{\pi}_p \\ \dot{\pi}_N \\ \dot{c}_N \end{pmatrix} = \underbrace{\begin{pmatrix} -k_p(1-\alpha\rho_{\pi_p}) & k_p(1-\alpha) & 0 \\ 0 & r & \frac{\delta}{c_N^{ss}} \\ c_N^{ss}\rho_{\pi p} & -c_N^{ss} & 0 \end{pmatrix}}_{W_s} \begin{pmatrix} \pi_p - \pi^{ss} \\ \pi_N - \pi^{ss} \\ c_N - c_N^{ss} \end{pmatrix} \tag{65}$$

where $\delta = \frac{\phi\left(c_N^{ss}\right)^{\frac{1}{\theta_N}}}{\gamma\theta_N^2} = \frac{(1+\phi)(1-\alpha)}{\gamma\theta_N} < 0$.

The following proposition summarizes the determinacy analysis for a pure backward-looking rule.

**Proposition 10** *Assume that* $R = \rho(\pi_p)$ *with* $\rho_{\pi_p} > 0$,

***a)*** *If* $\rho_{\pi_p} < 1$ *(a passive rule in terms of the weighted average of past CPI-inflation rates) then there exists a continuum of perfect foresight equilibria in which* $\{\pi_p, \pi_N, c_N\}$ *converge asymptotically to the steady state.*

***b)*** *if* $\rho_{\pi_p} > 1$ *(an active rule in terms of the weighted average of past CPI-inflation rates) then there exists a unique perfect foresight equilibrium in which* $\{\pi_p, \pi_N, c_N\}$ *converge asymptotically to the steady state.*

***Proof.*** *See Appendix.* ■

This last proposition shows that when the rule is a pure backward looking one, then multiple equilibria are possible only if the rule is passive. Hence an active rule with respect to the weighted average of past CPI-inflation rates always guarantees a unique equilibrium. It is important to notice that these results are independent of the degree of openness of the economy and on the weight that the government puts on past inflation rates.

Table 3 summarizes our results for rules with backward-looking and forward-looking elements.

**Table 3: Backward-Looking and Forward-Looking Rules**

*Interest Rate Smoothing*

$R = \rho(\pi, R_p) \qquad R_p = k_R \int_{-\infty}^{t} R(s)\mathrm{e}^{k_R(s-t)}ds \qquad k_R > 0,\ \rho_\pi > 0 \text{ and } \rho_R > 0$

| *Monetary Policy* | *Equilibrium* |
|---|---|
| $\rho_\pi > \frac{1}{\alpha}$ *or* $\rho_\pi + \rho_R < 1$ | *I* |
| $\rho_\pi > \frac{1}{\alpha}$ $1 < \rho_\pi < \frac{1}{\alpha}$ *or* $1 - \rho_R < \rho_\pi < 1$ | *D* |

*Forward-Looking Rules*

$R = \rho(\pi_f) \qquad \pi_f = k_f \int_{t}^{\infty} \pi(s)\mathrm{e}^{-k_f(s-t)}ds \qquad k_f > 0 \text{ and } \rho_{\pi_f} > 0$

| *Monetary Policy* | *Equilibrium* |
|---|---|
| $0 < \Omega < 1$ *and* $\{\rho_{\pi_f} < 1$ *or* $\rho^L < \rho_{\pi_f}\}$ | *I* |
| $0 < \Omega < 1$ *and* $1 < \rho_{\pi_f} < \rho^L$ | *D* |
| $\Omega > 1$ *and* $\rho_{\pi_f} > 0$ | *I* |

*Backward-Looking Rules*

$R = \rho(\pi_p) \qquad \pi_p = k_p \int_{-\infty}^{t} \pi(s)\mathrm{e}^{k_p(s-t)}ds \qquad k_p > 0 \text{ and } \rho_{\pi_p} > 0$

| *Monetary Policy* | *Equilibrium* |
|---|---|
| $\rho_{\pi_p} < 1$ | *I* |
| $\rho_{\pi_p} > 1$ | *D* |

*Note : The notation is D, determinate; I, indeterminate;* $\Omega = \frac{-(1+\phi)}{\theta_N \gamma r k_f}$;

$\rho^L = \frac{1}{\alpha} + \frac{\Omega(1-\alpha)^2}{2\alpha^2} + \frac{r}{2\alpha k_f} - \frac{\sqrt{(\Omega(1-\alpha)^2 k_f + \alpha r)^2 + 4\Omega k_f \alpha(1-\alpha)((1-\alpha)k_f + \alpha r)}}{2\alpha^2 k_f}$

*$\alpha$ is the degree of openness; $r$ is the world international interest rate;*
*$\theta_N, \phi$ and $\gamma$ are the labor income share and the degrees of monopolistic competition and price-stickiness in the non − traded sector respectively*

### 3.4 Two Extensions of the Basic Model

#### 3.4.1 The Utility Function

We consider a utility function in which the elasticity of substitution between traded and non-traded goods and the elasticity of intertemporal substitution are different than one, respectively.[31]

$$U_0 = \int_0^{\infty} \left[ A(c_{Tt}, c_{Nt}) + \frac{\nu \left(1 - h_{Tt} - h_{Nt}(j)\right)^{1+\xi}}{1+\xi} + \chi \log(m_t) - \frac{\gamma}{2} \left( \frac{\dot{P}_{Nt}(j)}{P_{Nt}(j)} - \pi_N^{ss} \right)^2 \right] \mathrm{e}^{-\beta t} dt \tag{66}$$

$$A(c_{Tt}, c_{Nt}) = \frac{\left\{ \left[ \alpha^{\frac{1}{\omega}} c_{Tt}^{\left(\frac{\omega-1}{\omega}\right)} + (1-\alpha)^{\frac{1}{\omega}} c_{Nt}^{\left(\frac{\omega-1}{\omega}\right)} \right]^{\frac{\omega}{\omega-1}} \right\}^{1-\sigma}}{1-\sigma} \tag{67}$$

where $\alpha,\ \beta \in (0,1)$, and $\sigma,\ \nu,\ \xi,\ \gamma,\ \omega,\ \chi > 0$.

We proceed as before in order to find the system of differential equations that govern the dynamics of this economy. However in this case it is not feasible to exploit the block structure that we exploited for equations (39), (40), (41) and (42). Therefore it is not possible to derive analytical results as we did before. Nevertheless we can assign values to the parameters of this economy and see graphically how our results of Propositions 1, 2 and 3 vary under this extended set-up.

As before we use Canada as the country of this exercise. Besides using the values of the parameters listed in Table 1 we borrow the following values of the parameters of other studies. From Mendoza (1991) we set $\nu = 1$ and $\xi = 0.455$. From Schmitt-Grohé and Uribe (2001) we set $\sigma = 2$ and from Mendoza (1995) we set $\omega = 0.74$.

First we consider the rule whose measure of inflation is the non-traded goods inflation that is $R = \rho(\pi_N)$. Figure 7 shows that for the utility function in (66) and for the parametrization used the results from Proposition 1 still hold. That is, an active rule with respect to the non-traded goods inflation rate guarantees a unique equilibrium while a passive rule leads to multiple equilibria.

Second we study a rule whose measure of inflation is the traded goods inflation, or given that there is perfect exchange rate pass-through, the nominal depreciation rate. That is $R = \rho(\epsilon)$. Figure 8 presents the results. From this figure we can deduce that the results from Proposition 2 are still valid. Regardless of how passive or active the rule is with respect to the nominal depreciation rate, the rule leads to real indeterminacy.

Finally we analyze an interest rate rule whose measure of inflation is the CPI-inflation, $R = \rho(\pi)$. Figures 9 and 10 present the results. The former corresponds to the case in which the elasticity of substitution between traded and non-traded goods is equal to $\omega = 0.74$. The latter corresponds

[31] In order to remove the distortionary effects of transactions money demand we still assume separability between the aggregator for consumption $A$ and money $m$. See Woodford (1998).

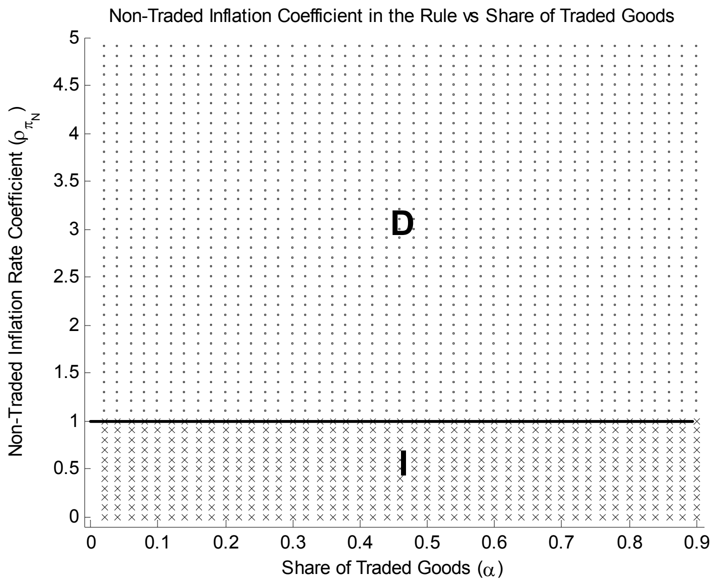


Figure 7: "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).


Figure 8: "I" stands for real indeterminacy (multiple equilibria).

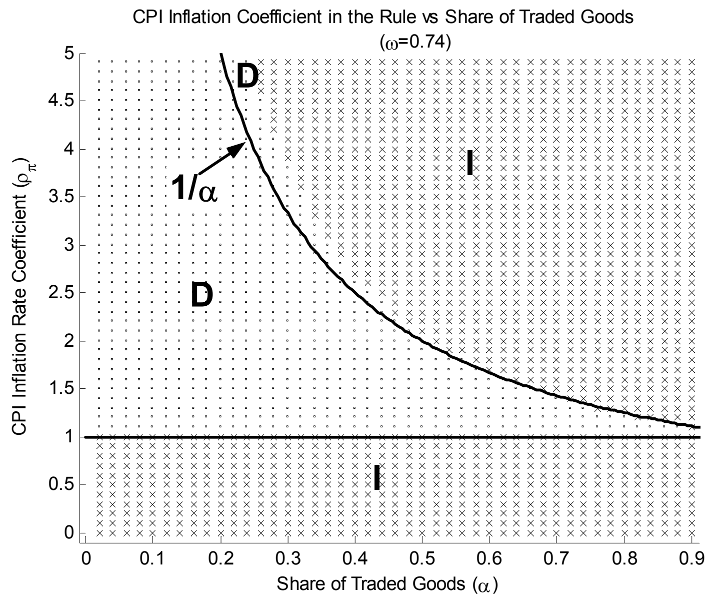


Figure 9: "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

to the case for which $\omega = 1.5$. A simple comparison of the two graphs suggests that the results of Proposition 3 hold in general terms. That is passive rules with respect to the CPI-inflation lead to multiple equilibria as part a) of Proposition 3 states. In addition, although it is true that the degree of openness of the economy matters for the determinacy of equilibrium of active rules, there are some slight differences with respect to the results of part b) and c) of Proposition 3. For elasticities of substitution between traded and non-traded goods that are less than one, $\omega < 1$, the condition that $\rho_\pi > \frac{1}{\alpha}$ becomes a necessary condition, but not sufficient, for active rules to induce real indeterminacy. On the other hand for $\omega > 1$, the condition that $\rho_\pi > \frac{1}{\alpha}$ is still a sufficient condition for multiple equilibria but note that in this case the region of real indeterminacy for active rules expands in comparison with the same region for active rules when $\omega < 1$. Besides these differences the message of this analysis is basically the same of Proposition 3. The more open the economy is the more likely is that an active rule with respect to the CPI-inflation may induce aggregate instability in the economy by generating multiple equilibria.

### 3.4.2 Distributional Costs and Imperfect Exchange Rate Pass-through

The results that we derived in the study of an interest rate rule that responds to the CPI-inflation are in some sense driven by the assumption that PPP holds for traded goods. The reason is that this assumption implies that there is a perfect pass-through from changes in the nominal exchange

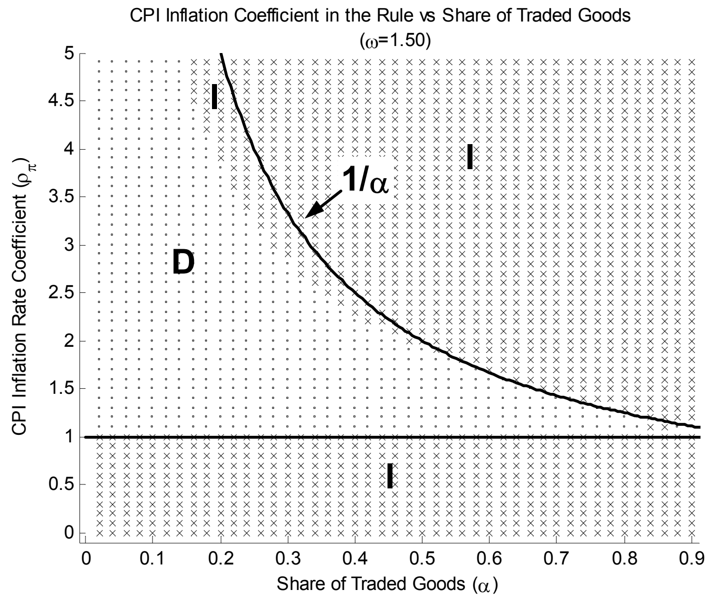


Figure 10: "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

rate to changes in the price of traded goods. And as it was shown when the measure of inflation is the traded goods inflation rate or the nominal depreciation rate then multiple equilibria arise regardless of how active or passive the rules is. But the CPI-inflation ($\pi$) is a weighted average of the nominal depreciation rate ($\epsilon$) and the non-traded goods inflation rate ($\pi_N$), where the weights are related to the degree of openness of the economy ($\alpha$). That is $\pi = \alpha\epsilon + (1-\alpha)\pi_N$. Therefore as the degree of openness of the economy ($\alpha$) increases, the CPI-inflation ($\pi$) resembles more the nominal depreciation rate ($\epsilon$) and therefore it is more likely that an active rule with respect to the CPI-inflation will deliver real indeterminacy in our model.

In this part of the paper we relax the assumption about PPP for traded goods. Relaxing this assumption will allow us to model the case of imperfect exchange rate pass-through. To do so we follow Burnstein, Neves and Rebelo (2003).[32] It assumes that the traded good needs to be combined

[32] Monacelli (1999), Devereux (2001) and Smets and Wouters (2002) follow a different approach. They assume that foreign suppliers may choose a pricing policy that stabilizes the prices of imports in terms of the local currency of the small open economy. Domestic consumers of the small open economy however take the local currency price of imported goods as given. This approach yields a Phillips curve for the inflation of the traded goods similar to the the one in (26) for the inflation of non-traded goods. In this sense the price of the traded good is considered sticky. Since we still want to keep the price flexibility in the traded sector and price-stickiness in the non-traded sector we follow the approach of distributional costs. This will facilitate comparisons with the results in Propositions 1, 2 and 3.

with some non-traded distribution services before it is consumed.[33] Assume that to consume one unit of the traded good it is required $\eta$ units of the non-traded composite good. Let $\tilde{P}_T$ and $P_T$ be the prices in the domestic currency of the small open economy that producers of traded goods receive and that consumers pay, respectively. Hence the consumer price of the traded good is simply

$$P_T = \tilde{P}_T + \eta P_N \tag{68}$$

To simplify the analysis we assume that PPP holds for the producers of the traded goods and we normalize the foreign price of the traded good to one ($\tilde{P}_T^* = 1$). Hence

$$\tilde{P}_T = E\tilde{P}_T^* = E \tag{69}$$

Using (68) and (69) and defining $e = E/P_N$ we can rewrite the budget constraint of the household-firm unit (10) as

$$\dot{b} = rb + Ra + \tau + y_T + \frac{P_N(j)}{P_N}\frac{y_N(j)}{e} - \left(1 + \frac{\eta}{e}\right)c_T - \frac{c_N}{e} - \epsilon\,(m + a) - (\dot{m} + \dot{a}) - z \tag{70}$$

We still assume that the agent maximizes (1) subject to (2) and the rest of the constraints and that the government behaves as we specified in the simple set-up of our model. The important difference is that under distributional costs the equilibrium condition for the non-traded good implies that $y_N = Y^d = c_N + \eta c_T$.

Note that the introduction of distributional costs is a way to model the imperfect exchange rate pass-through. To see this use (68) and (69) to derive the inflation of the traded good for the price paid by consumers in the small open economy as

$$\pi_T = \left(\frac{e}{e+\eta}\right)\epsilon + \left(\frac{\eta}{e+\eta}\right)\pi_N \tag{71}$$

Therefore if $\eta = 0$ then we have perfect pass-through of the nominal depreciation rate into the traded good inflation rate, $\pi_T = \epsilon$. This is the case that we already studied. But if $\eta > 0$ then we obtain imperfect pass-through in this model and it is measured by $\frac{d\pi_T}{d\epsilon} = \left(\frac{e}{e+\eta}\right) > 0$. Moreover since we are still using the aggregator function for consumption described in (2) it is straightforward to derive the CPI-inflation as:

$$\pi = \alpha\left(\frac{e}{e+\eta}\right)\epsilon + \left(\frac{(1-\alpha)e+\eta}{e+\eta}\right)\pi_N \tag{72}$$

We proceed as before in order to find the system of differential equations that govern the dynamics of this economy. However in this case it is not possible to use the block structure that we exploited for equations (39), (40), (41) and (42). Hence it is not feasible to derive analytical

[33] Corsetti and Dedola (2002) and Corsetti, Dedola and Leduc (2003) follow this approach.

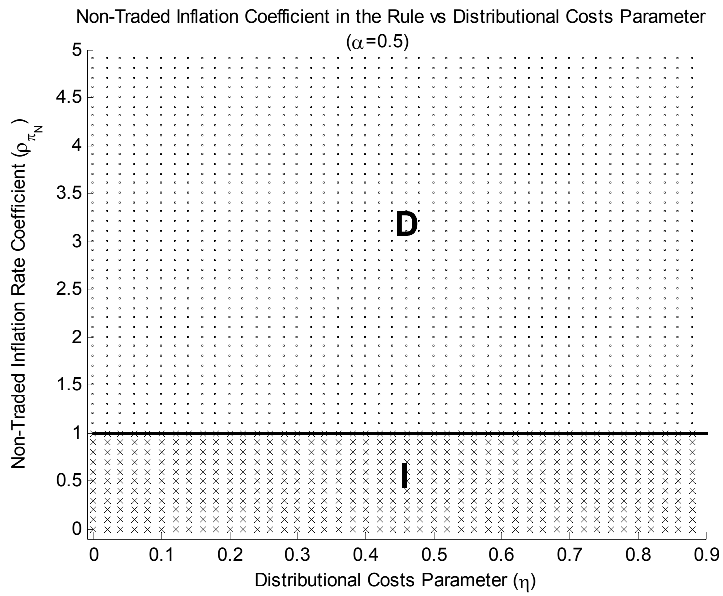


Figure 11: "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

results as we did before. Nevertheless we can assign values to the parameters of this economy and see graphically how our results of Propositions 1, 2 and 3 vary under this extended set-up. We use the aforementioned parametrization and follow Corsetti and Dedola (2002) in setting $\eta = 0.5$ when we do not vary this parameter. Moreover following Devereux (2001) we set the degree of openness of the economy to $\alpha = 0.5$ when we do not vary this parameter.

First we consider the rule whose measure of inflation is the non-traded goods inflation that is $R = \rho(\pi_N)$. Figure 11 shows the results for this rule when $\eta$ varies. In general the results of Proposition 1 are still valid. That is if the target of inflation is the non-traded goods inflation rate, active rules will avoid the possibility of multiple equilibria even if there is imperfect pass-through.

Second we analyze the rule whose measure of inflation corresponds to the traded goods inflation rate. That is $R = \rho(\pi_T)$. Note that in this case due to the existence of the distributional costs there is an imperfect exchange rate pass-through and therefore the traded goods inflation rate will not coincide with the nominal depreciation rate. The results of Proposition 2 still hold with imperfect exchange rate pass-through and when the rule is defined in terms of the nominal depreciation rate.[34] However Figure 12 shows that once we consider the traded goods inflation rate instead of the nominal depreciation rate then the degree of openness of the economy matters for the determinacy of equilibrium. In particular, given the imperfect exchange rate pass-through, the more open the

[34] These results are available from the author upon request.

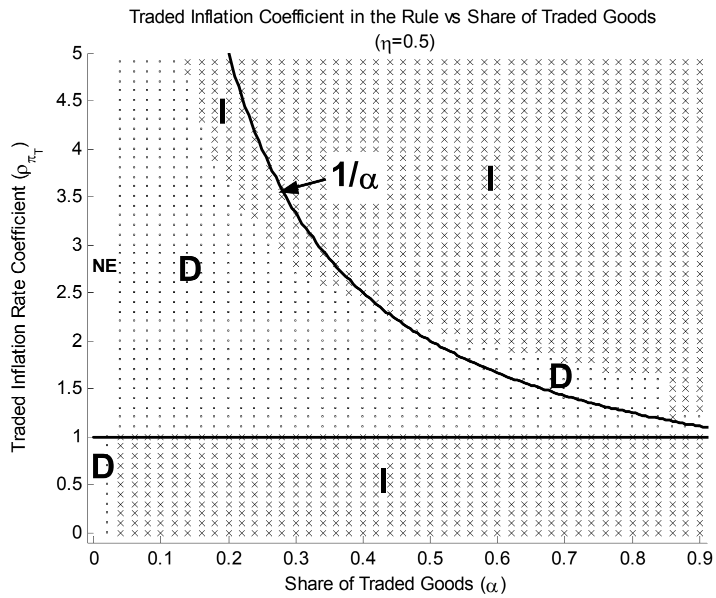


Figure 12: "NE" stands for non-existence of equilibrium; "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

economy is, the more likely is that a rule that responds to the traded goods inflation rate will lead to real indeterminacy.

To see the importance of the imperfect exchange rate pass-through in this analysis we can set the degree of openness of the economy to $\alpha = 0.5$, and vary the parameter of distributional costs $\eta$. The results are presented in Figure 13. As expected when there is a perfect exchange rate pass-through ($\eta = 0$) and the government targets the traded inflation rate, then multiple equilibria arise regardless of how responsive the rule is with respect to this measure of inflation. This is because in this case the traded goods inflation rate coincides with the nominal depreciation rate and therefore the results of Proposition 2 apply. However if the distributional costs increase, that is if there is imperfect exchange rate pass-through, then following an active rule with respect to the traded goods inflation rate may actually lead to real determinacy. The higher the imperfect exchange rate pass-through is then the more likely is that this rule will lead to a unique equilibrium. This result must be clear once we recall equation (71) that describes the inflation of the non-traded goods as an average of the nominal depreciation rate and the non-traded goods inflation rate. The weights in this equation are related to the parameter $\eta$ of the distributional costs. Hence the higher is $\eta$ the lower is the weight on the nominal depreciation rate and the higher is the weight on the non-traded inflation rate. But from Proposition 1 and Figure 11, active rules with respect to the non-traded inflation guarantee a unique equilibrium. Therefore this effect prevails when $\eta$ is high

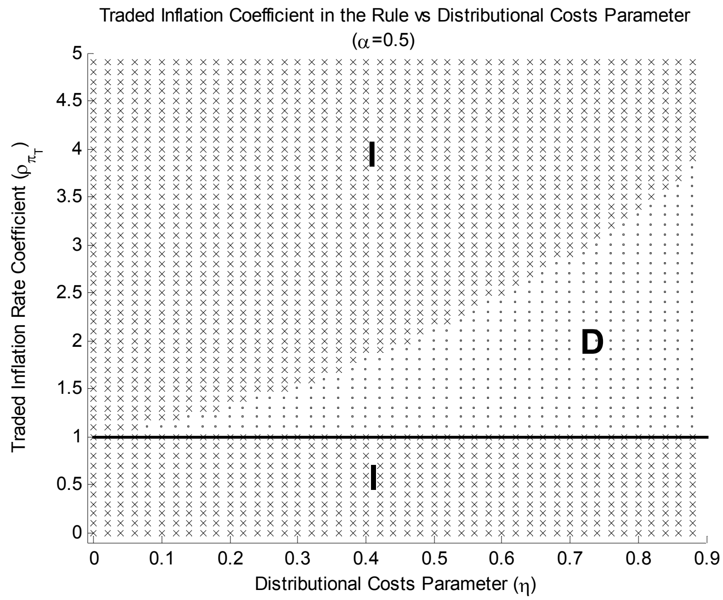


Figure 13: "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

and the measure of the inflation of the rule is the traded goods inflation rate.

Finally we study the rule whose measure of inflation corresponds to the CPI-inflation. That is $R = \rho(\pi)$. Figure 14 summarizes the results. From this figure it is possible to infer that some of the results of Proposition 3 still hold under imperfect exchange rate pass-through. In particular it is true that passive rules still lead to real indeterminacy and that the degree of openness of the economy matters for the determinacy of equilibrium analysis. However notice that in this case the condition that $\rho_\pi > \frac{1}{\alpha}$ is not longer a sufficient condition for a rule to induce multiple equilibria but instead it is a necessary condition. The results of Figure 14 can be understood if we pursue the analysis of fixing the degree of openness of the economy ($\alpha = 0.5$), and vary the parameter of distributional costs ($\eta$). The results are presented in Figure 15. From this figure we can infer that the higher the imperfect exchange rate pass-through is, the more likely is that an active rule with respect to the CPI-inflation rate will lead to a unique equilibrium. To understand the results in Figure 14 and 15 it is sufficient to recall equation (72) that represents the CPI-inflation rate as a weighted average of the nominal depreciation rate and the non-traded goods inflation rate. The weights are clearly functions of the degree of openness of the economy $\alpha$ and the parameter of distributional costs $\eta$. Moreover remember that the possibility of real indeterminacy under active rules with respect to the CPI-inflation stems from the direct effect that the nominal depreciation rate has on the CPI-inflation rate. The more open the economy is (that is the greater $\alpha$ is) and the more perfect the pass-through is (the lower $\eta$ is), then the greater this direct aforementioned effect is. But the greater this effect is the higher the possibility of having multiple equilibria under active rules. In the extreme case when the degree of openness $\alpha$ is close to 1, and there is perfect exchange rate pass-through, $\eta = 0$, the CPI-inflation rate coincides with the nominal depreciation rate. Then the results from Proposition 2 apply. That is multiple equilibria arise under active rules (see Figure 1). On the other hand, in the extreme case when the degree of openness $\alpha$ is close to 0, and there is a very high imperfect pass-through, the CPI-inflation rate tends to the non-traded goods inflation rate and then we recover the results from Proposition 1.

To some extent the previous analysis confirms the proposals by Devereux and Lane (2001). They say that if there is a high exchange rate pass-through, a policy of non-traded goods inflation targeting does better stabilizing the economy than a policy of CPI-inflation targeting. Our results are derived from a different approach. We have done a determinacy of equilibrium analysis and we have arrived to the conclusion that in order to avoid aggregate instability by generating multiple equilibria, the government should target the non-traded inflation rate. We summarize these results in the following Proposition.

**Proposition 11** *Even under imperfect exchange rate pass-through the degree of openness of the economy matters for the determinacy of equilibrium of active interest rate rules with respect to either the CPI-inflation rate or the traded goods inflation rate. The more open the economy is, the more likely is that these rules will lead to multiple equilibria. On the other hand, a rule that responds actively to the non-traded goods inflation avoids the presence of multiple equilibria.*

## 4 Concluding Remarks

In this paper we isolate and identify conditions that are sufficient to ensure that interest-rate feedback rules do not induce aggregate instability by generating multiple equilibria in the small open economy. We show that when the government follows an interest rate rule, conditions that lead to real indeterminacy depend not only on the type of monetary policy, active or passive, but also on the measure of inflation to which the government responds, on the degree of openness of the economy and on the degree of exchange rate pass-through.

Most of our determinacy of equilibrium results are driven by the fact that in our model an interest rate rule that responds solely to the nominal depreciation rate always leads to multiple equilibria. Whereas if the only argument of the rule is the non-traded goods inflation rate, then active rules guarantee a unique equilibrium. To the extent that the CPI-inflation rate is a weighted average of the traded goods inflation rate (that is affected by the nominal depreciation rate) and of the non-traded goods inflation, it is clear that active rules with respect to the CPI-inflation rate

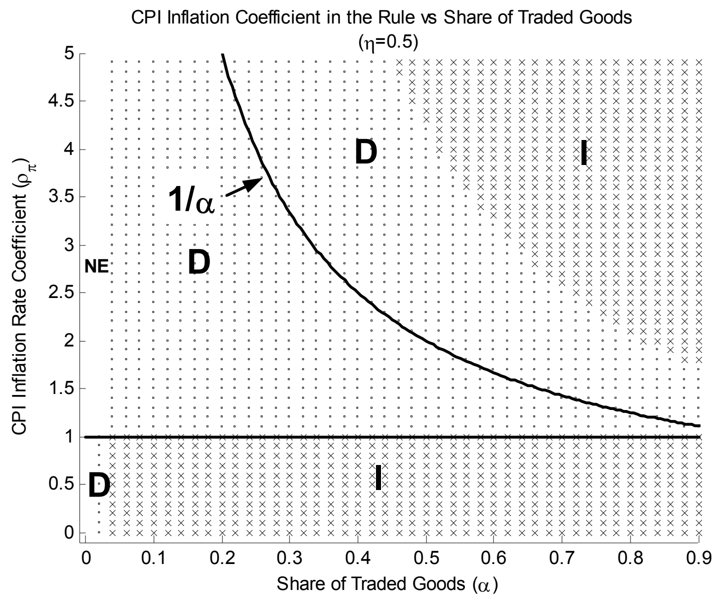


Figure 14: "NE" stands for non-existence of equilibrium; "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

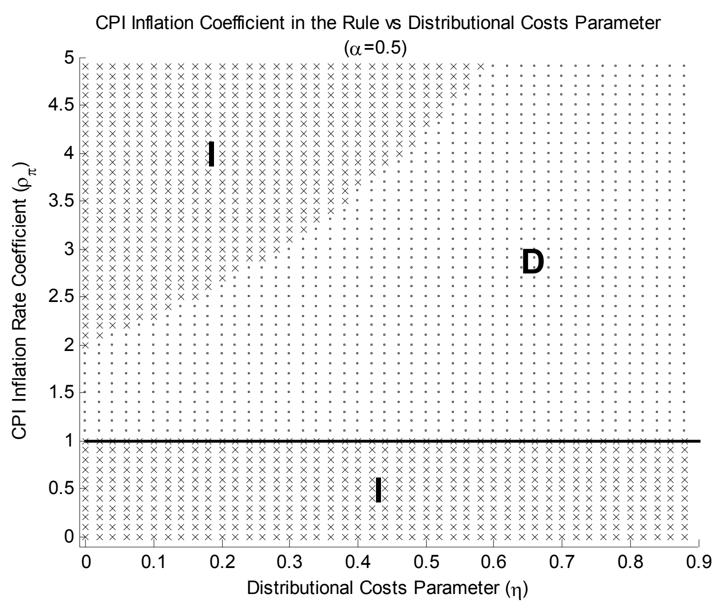


Figure 15: "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium).

may lead to real indeterminacy. In particular, depending on the degree of openness, active rules with respect to the CPI-inflation may induce multiple equilibria. This result is very important because it calls into question the interpretation given to some of the estimations of interest rate rules in small open economies.[35] It points out that active rules do not necessarily induce stability for open economies.

Our analysis suggests that the measure of inflation that should be taken into account in the design of a rule for the small open economy is the non-traded goods inflation rate or at least a measure of inflation that is not heavily affected by the nominal depreciation rate. Since in our model the non-traded sector has sticky prices whereas the traded sector has flexible prices, our results are similar to those of Aoki (2001) and Mankiw and Reis (2002) in the sense that the government should target the inflation of the sector that has (more) sticky prices.

The degree of openness is still a fundamental factor in the local equilibrium analysis for extended rules that include combinations of the CPI-inflation rate, the output gap, the nominal depreciation rate, the real exchange rate and/or past interest rates. As a by-product of this analysis we find that "fear of floating" governments that follow a rule that responds to both the CPI-inflation rate and the nominal depreciation rate may actually be destabilizing their economies.

For rules that depend on expected future CPI-inflation rates we find that the conditions for determinacy not only depend on the degree of openness but also on the weight that the monetary authority puts on these inflation rates. If the central bank puts a high weight on distant future expected CPI-inflation rates then the rule always leads to multiple equilibria. In contrast, a backward-looking interest-rate feedback rule always guarantees a unique equilibrium if the rule is active with respect to the weighted average of past CPI-inflation rates.

Finally we want to discuss briefly two of our assumptions and the consequences associated with them. First, we assumed a Ricardian fiscal policy. Following the analysis of Benhabib et al. (2001a) it is straightforward to show that rules that lead to multiple equilibria under Ricardian fiscal policies may actually lead to a unique equilibrium under Non-Ricardian fiscal policies.

Second the results presented in this paper were derived from a local determinacy of equilibrium analysis. However once a zero bound for the nominal interest rate is considered and a global analysis is pursued, it is possible to show that rules that respond exclusively to the CPI-inflation may also induce a special type of endogenous fluctuations. In fact Airaudo and Zanna (2003a) show that the more open the economy is the more likely it is that a contemporaneous rule will drive the economy into a liquidity trap. On the other hand they find that the more closed the economy is, the more likely it is that the same rule will lead to cycles and chaotic dynamics around the inflation target. It is important to observe that this does not imply that the possibility of cycles only arises under global analysis. Even under local analysis cycles may arise as a consequence of "Hopf bifurcations"

[35] See Clarida et al. (1998) among others.

as Airaudo and Zanna (2003b) show for forward-looking rules in the small open economy.

# 5 Appendix

## 5.1 Proofs of Propositions

In the following propositions we apply repeatedly the results from Blanchard and Kahn (1980) and Buiter (1984).

**Proof of Proposition 1**

**Proof.** First, if $\rho_\pi = \rho_\epsilon = \rho_e = \rho_{\dot{e}/e} = \rho_y = 0$ then using the expressions (48) and (49) we derive that

$$Trace(J_s) = r > 0 \qquad Det(J_s) = \frac{\phi\,(c_N^{ss})^{\frac{1}{\theta_N}}}{\gamma\theta_N^2}(1-\rho_{\pi_N})$$

with $\phi < 0$. Second for a) if $\rho_{\pi_N} < 1$ then we can deduce that $Det(J_s) < 0$ implying that $J_s$ has one eigenvalue with a negative real part and one eigenvalue with a positive real part. Given that there are two jump variables $(\pi_N,\, c_N)$, the number of jump variables is greater than the number of explosive roots. Hence there is real indeterminacy.

For b) if $\rho_{\pi_N} > 1$ then $Det(J_s) > 0$ and since $Trace(J_s) > 0$ we can conclude that there are two roots with positive real parts and therefore there is real determinacy. ■

**Proof of Proposition 2**

**Proof.** If $\rho_\pi = \rho_{\pi_N} = \rho_e = \rho_{\dot{e}/e} = \rho_y = 0$ then using expression (49) we derive that

$$Det(J_s) = \frac{\phi\,(c_N^{ss})^{\frac{1}{\theta_N}}}{\gamma\theta_N^2} < 0$$

with $\phi < 0$. Since $Det(J_s) < 0$ then $J_s$ has one eigenvalue with a negative real part and one eigenvalue with a positive real part. Given that there are two jump variables $(\pi_N,\, c_N)$, the number of jump variables is greater than the number of explosive roots. Hence there are multiple equilibria.
■

**Proof of Proposition 3**

**Proof.** First, if $\rho_\epsilon = \rho_{\pi_N} = \rho_e = \rho_{\dot{e}/e} = \rho_y = 0$ then using the expressions (48) and (49) we derive that

$$Trace(J_s) = r > 0 \qquad Det(J_s) = \frac{\phi\,(c_N^{ss})^{\frac{1}{\theta_N}}}{\gamma\theta_N^2\alpha}\frac{(1-\rho_\pi)}{(\frac{1}{\alpha}-\rho_\pi)}$$

Second it should be remembered that $0 < \alpha < 1$ and $\phi < 0$. For a) and b) if either $\rho_\pi < 1$ (and therefore $\rho_\pi < \frac{1}{\alpha}$) or $1 < \frac{1}{\alpha} < \rho_\pi$ then we have that in both cases $Det(J_s) < 0$ implying that $J_s$ has one eigenvalue with a negative real part and one eigenvalue with a positive real part. Given that there are two jump variables $(\pi_N,\, c_N)$, the number of jump variables is greater than the number of explosive roots. Hence near the steady state there exists an infinite number of perfect foresight equilibria converging to the steady state.

For part c) it should be observed that if $1 < \rho_\pi < \frac{1}{\alpha}$ then we can infer that $Det(J_s) > 0$. This result in tandem with $Trace(J_s) > 0$ allows us to conclude that the two eigenvalues have positive real parts. Thus the number of jump variables is equal to the number of explosive roots. Thus there exists a unique perfect foresight equilibrium. ■

**Proof of Proposition 4**

**Proof.** For a) if $\rho_\pi > 0$, $\rho_\epsilon > 0$, $\rho_e > 0$, $\rho_{\dot{e}/e} > 0$, and $\rho_y > 0$ then using expression (49) we can derive that

$$Det(J_s) = r\frac{\rho_y(1-\alpha)y^{ss} + \rho_e e^{ss}}{1 - \alpha\rho_\pi - \rho_\epsilon - \rho_{\dot{e}/e}} + \frac{\phi\left(c_N^{ss}\right)^{\frac{1}{\theta_N}}}{\gamma\theta_N^2}\frac{(1-\rho_\pi-\rho_\epsilon)}{(1-\alpha\rho_\pi-\rho_\epsilon-\rho_{\dot{e}/e})}$$

Since $1 < \frac{1}{\alpha} < \rho_\pi$ then we can deduce that $Det(J_s) < 0$. This implies that $J_s$ has one eigenvalue with a negative real part and one eigenvalue with a positive real part. Given that there are two jump variables ($\pi_N$, $c_N$), the number of jump variables is greater than the number of explosive roots. Hence real indeterminacy follows. ■

**Proof of Proposition 5**

**Proof.** First, if $\rho_{\pi_N} = \rho_e = \rho_{\dot{e}/e} = \rho_y = 0$ then using the expressions (48) and (49) we derive that

$$Trace(J_s) = r > 0 \qquad Det(J_s) = \frac{\phi\left(c_N^{ss}\right)^{\frac{1}{\theta_N}}}{\gamma\theta_N^2}\frac{(1-\rho_\pi-\rho_\epsilon)}{(1-\alpha\rho_\pi-\rho_\epsilon)}$$

Second, for a) given that $0 < \alpha < 1$, if either $\rho_\pi + \rho_\epsilon < 1$ or $1 < \alpha\rho_\pi + \rho_\epsilon$ then $\alpha\rho_\pi + \rho_\epsilon < 1$ or $1 < \rho_\pi + \rho_\epsilon$. Under both assumptions it is clear that $Det(J_s) < 0$ Thus $J_s$ has one eigenvalue with a negative real part and one eigenvalue with a positive real part. Therefore the number of jump variables, $\{\pi_N, c_N\}$, is greater than the number of explosive roots implying that there are multiple equilibria.

For b) if $\alpha\rho_\pi + \rho_\epsilon < 1 < \rho_\pi + \rho_\epsilon$ then we can infer that $Det(J_s) > 0$. This result in tandem with $Trace(J_s) > 0$ allows us to conclude that there are two eigenvalues with positive real parts. Thus the number of jump variables,$\{\pi_N, c_N\}$ , is equal to the number of explosive roots. Hence real determinacy follows. ■

**Proof of Proposition 6**

**Proof.** If $\rho_\pi = \rho_e = \rho_y = \rho_{\dot{e}/e} = 0$ then using the expressions (48) and (49) we derive that

$$Trace(J_s) = r > 0 \qquad Det(J_s) = \frac{\phi\left(c_N^{ss}\right)^{\frac{1}{\theta_N}}}{\gamma\theta_N^2}\frac{\left(1-\rho_{\pi_N}-\rho_\epsilon\right)}{(1-\rho_\epsilon)}$$

For a) if either $\rho_{\pi_N} + \rho_\epsilon < 1$ (which implies $\rho_\epsilon < 1$ since $\rho_{\pi_N} > 0$) or $\rho_\epsilon > 1$ (which implies that $1 < \rho_{\pi_N} + \rho_\epsilon$ since $\rho_\epsilon > 0$) then $Det(J_s) < 0$. Thus $J_s$ has one eigenvalue with a negative real part and one eigenvalue with a positive real part. Therefore the number of jump variables, $\{\pi_N, c_N\}$, is greater than the number of explosive roots implying that there are multiple equilibria.

For b) if $1 < \rho_{\pi_N} + \rho_\epsilon$ and $\rho_\epsilon < 1$ then $Det(J_s) > 0$. This result in tandem with $Trace(J_s) > 0$ allows us to conclude that there are two roots with positive real parts. Thus the number of jump variables, $\{\pi_N, c_N\}$, is equal to the number of explosive roots which implies that there is a unique equilibrium. ■

**Proof of Proposition 7**

**Proof.** To prove this proposition it is useful to derive expressions for the trace, the sum of the $2 \times 2$ principal minors and the determinant of the matrix $N_s$ in (62). That is

$$Trace(N_s) = \frac{r(1-\alpha\rho_\pi) - k_R\,(1-\rho_R-\alpha\rho_\pi)}{(1-\alpha\rho_\pi)} \tag{73}$$

$$S_2(N_s) = \frac{\delta\,(1-\rho_\pi) - rk_R\,(1-\rho_R-\alpha\rho_\pi)}{(1-\alpha\rho_\pi)} \tag{74}$$

$$Det(N_s) = -\frac{k_R\delta\,(1-\rho_\pi-\rho_R)}{(1-\alpha\rho_\pi)} \tag{75}$$

where

$$\delta = \frac{\phi\,(c_N^{ss})^{\frac{1}{\theta_N}}}{\gamma\theta_N^2} = \frac{(1+\phi)(1-\alpha)}{\gamma\theta_N} < 0$$

In addition it is important to remember that $Trace(N_s) = \omega_1 + \omega_2 + \omega_3$, $S_2(N_s) = \omega_1\omega_2 + \omega_1\omega_3 +$ $+\omega_2\omega_3$ and $Det(N_s) = \omega_1\omega_2\omega_3$, where the $\omega_i$ 's correspond to the eigenvalues for $N_s$ (See theorem 1.2.12 from Horn and Johnson (1985)).

By assumption $\rho_\pi > 0$ and $\rho_R > 0$. For a), notice that $\rho_\pi > \frac{1}{\alpha} > 1$. Then using expressions (74) and (75) we conclude that $S_2(N_s) < 0$ and $Det(N_s) > 0$. On the other hand, $\rho_\pi + \rho_R < 1$ implies that $\rho_\pi < 1$ and since $0 < \alpha < 1$ it also implies that $\alpha\rho_\pi + \rho_R < 1$ which in turn implies that $\alpha\rho_\pi < 1$. Therefore if $\rho_\pi + \rho_R < 1$ then using expressions (74) and (75) we can infer that $S_2(N_s) < 0$ and $Det(N_s) > 0$. Thus if either $\rho_\pi > \frac{1}{\alpha}$ or $\rho_\pi + \rho_R < 1$ then $S_2(N_s) < 0$ and $Det(N_s) > 0$. This in turn implies by Theorem 1.2.12 from Horn and Johnson (1985) that $N_s$ has one eigenvalue with a positive real part and two eigenvalues with negative real parts. Given that there are two jump variables ($\pi_N$, $c_N$), the number of jump variables is greater than the number of explosive roots. Applying the results of Blanchard and Kahn (1980) and Buiter (1984) it follows that there is real indeterminacy.

For b) note that the sufficient condition given in the statement is equivalent to $\rho_\pi < \frac{1}{\alpha}$ and $1 - \rho_R < \rho_\pi$. It is necessary to consider two possibilities: $k_R > r$ and $k_R < r$.

For $k_R > r$ we divide the region in the positive plane $\rho_\pi$ vs $\rho_R$ that is between $\rho_\pi < \frac{1}{\alpha}$ and $1 - \rho_R < \rho_\pi$ in three exclusive subregions: subregion I for which $\rho_\pi > \frac{1}{\alpha} - \frac{1}{\alpha}\left(\frac{k_R}{k_R - r}\right)\rho_R$, subregion II for which $\frac{rk_R - \delta}{\alpha rk_R - \delta} - \left(\frac{rk_R}{\alpha rk_R - \delta}\right)\rho_R < \rho_\pi < \frac{1}{\alpha} - \frac{1}{\alpha}\left(\frac{k_R}{k_R - r}\right)\rho_R$ and subregion III for which

$\rho_\pi < \frac{1}{\alpha} - \frac{1}{\alpha}\left(\frac{k_R}{k_R - r}\right)\rho_R$ and $\rho_\pi < \frac{rk_R-\delta}{\alpha r k_R - \delta} - \left(\frac{rk_R}{\alpha r k_R - \delta}\right)\rho_R$, where $\delta = \frac{\phi\left(c_N^{ss}\right)^{\frac{1}{\theta_N}}}{\theta_N^2 \gamma} = \frac{(1+\phi)(1-\alpha)}{\theta_N \gamma} < 0$. See Figure 5 where $\rho_\pi^{f1} = \frac{1}{\alpha} - \frac{1}{\alpha}\left(\frac{k_R}{k_R - r}\right)\rho_R$ and $\rho_\pi^{f2} = \frac{rk_R-\delta}{\alpha r k_R - \delta} - \left(\frac{rk_R}{\alpha r k_R - \delta}\right)\rho_R$. It is straightforward to prove that these two boundaries intersect in a point $(\rho_R^*, \rho_\pi^*)$ such that $\rho_\pi^* > 1$.

For all the three subregions note that if $\rho_\pi < \frac{1}{\alpha}$ and $1 < \rho_\pi + \rho_R$ then using (75) we derive that $Det(N_s) < 0$.

For subregion I, note that $\rho_\pi > \frac{1}{\alpha} - \frac{1}{\alpha}\left(\frac{k_R}{k_R - r}\right)\rho_R$ implies that $r(1-\alpha\rho_\pi) - k_R(1-\rho_R-\alpha\rho_\pi) > 0$ that in tandem with $\rho_\pi < \frac{1}{\alpha}$ imply, from (73), that $Trace(N_s) > 0$. Therefore we have that for this subregion $Det(N_s) < 0$ and $Trace(N_s) > 0$. Thus applying Theorem 1.2.12 from Horn and Johnson (1985) we can conclude that $N_s$ has two eigenvalues with positive real parts and one eigenvalue with a negative real part.

For subregion II, $\rho_\pi < \frac{1}{\alpha} - \frac{1}{\alpha}\left(\frac{k_R}{k_R - r}\right)\rho_R$ implies that $r(1-\alpha\rho_\pi) - k_R(1-\rho_R-\alpha\rho_\pi) < 0$. This result in tandem with $\rho_\pi < \frac{1}{\alpha}$ imply, from (73), that $Trace(N_s) < 0$. In addition note that $\frac{rk_R-\delta}{\alpha r k_R - \delta} - \left(\frac{rk_R}{\alpha r k_R - \delta}\right)\rho_R < \rho_\pi$ means that $\delta\,(1-\rho_\pi) - rk_R\,(1-\rho_R-\alpha\rho_\pi) > 0$, that together with $\rho_\pi < \frac{1}{\alpha}$ and (74) allow us to conclude that $S_2(N_s) > 0$. Now we invoke the Theorem of Routh-Hurwicz.[36] This theorem states that the number of roots of $N_s$ with positive real parts is equal to the number of variations of sign in the scheme

$$1 \qquad -Trace(N_s) \qquad \frac{S_2(N_s)Trace(N_s) - Det(N_s)}{Trace(N_s)} \qquad -Det(N_s) \tag{76}$$

Note that $Det(N_s)$ can be written as $Det(N_s) = -k_R S_2(N_s) - \frac{rk_R^2(1-\rho_\pi-\rho_R)}{(1-\alpha\rho_\pi)} + \frac{k_R\delta\rho_R}{(1-\alpha\rho_\pi)}$. Using this expression and (73) we can derive that

$$\frac{S_2(N_s)Trace(N_s) - Det(N_s)}{Trace(N_s)} = \frac{S_2(N_s)\left(r(1-\alpha\rho_\pi) + k_R\rho_R\right) + rk_R^2\left(1-\alpha\rho_\pi-\rho_R\right) - \delta k_R \rho_R}{(1-\alpha\rho_\pi)Trace(N_s)}$$

Notice that the numerator of this expression is positive since $\rho_\pi < \frac{1}{\alpha}$ and $S_2(N_s) > 0$ and $\rho_R > 0$ and $\delta < 0$ and $\rho_\pi < \frac{1}{\alpha} - \frac{1}{\alpha}\left(\frac{k_R}{k_R - r}\right)\rho_R < \frac{1}{\alpha} - \frac{1}{\alpha}\rho_R$; while its denominator is negative given that $Trace(N_s) < 0$ and $\rho_\pi < \frac{1}{\alpha}$. This means that $\frac{S_2(N_s)Trace(N_s)-Det(N_s)}{Trace(N_s)} < 0$. This result in tandem with $Det(N_s) < 0$ and $Trace(N_s) < 0$ imply, by the Theorem of Routh-Hurwicz, that $N_s$ has two eigenvalues with positive real parts and one eigenvalue with a negative real part in the subregion II.

Finally in the subregion III we have that $\frac{rk_R-\delta}{\alpha r k_R - \delta} - \left(\frac{rk_R}{\alpha r k_R - \delta}\right)\rho_R > \rho_\pi$ which implies that $\delta\,(1-\rho_\pi) - rk_R\,(1-\rho_R-\alpha\rho_\pi) < 0$. This inequality together with $\rho_\pi < \frac{1}{\alpha}$ and expression (74) lead to infer that $S_2(N_s) < 0$. But for this subregion it is still true that $Det(N_s) < 0$. Hence applying Theorem 1.2.12 from Horn and Johnson (1985) we can conclude that $N_s$ has two eigenvalues with positive real parts and one eigenvalue with a negative real part.

[36] See Gantmacher (1960) for the Theorem of Routh-Hurwicz.

For $k_R < r$ since $1 - \rho_R < \rho_\pi < \frac{1}{\alpha}$ then using (75) we derive that $Det(N_s) < 0$. Moreover since $k_R < r$ and $\rho_\pi < \frac{1}{\alpha}$ then $\rho_\pi < \frac{1}{\alpha} - \frac{1}{\alpha}\left(\frac{k_R}{k_R - r}\right)\rho_R$ which in turn implies that $r(1-\alpha\rho_\pi) - k_R(1 - \rho_R - \alpha\rho_\pi) < 0$. But this result in tandem with $\rho_\pi < \frac{1}{\alpha}$ and (73), allow us to conclude that $Trace(N_s) > 0$. Thus applying Theorem 1.2.12 from Horn and Johnson (1985) we can conclude that $N_s$ has two eigenvalues with positive real parts and one eigenvalue with a negative real part.

Since for either $k_R > r$ (within the three subregions I, II and III) or for $k_R < r$ we have that $N_s$ has two eigenvalues with positive real parts and one eigenvalue with a negative real part, then the number of jump variables is equal to the number of explosive roots ($\pi_N$, $c_N$). Once more we apply the results of Blanchard and Kahn (1980) and Buiter (1984) to state that in this case there exists a unique perfect foresight equilibrium. ■

**Proof of Proposition 8**

**Proof.** To prove this proposition it is useful to derive expressions for the trace, the sum of the $2 \times 2$ principal minors and the determinant of the matrix $L_s$ in (64)

$$Trace(L_s) = r + k_f(1 - \alpha\rho_{\pi_f}) \tag{77}$$

$$S_2(L_s) = k_f r\alpha\left(\frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha} - \rho_{\pi_f}\right) \tag{78}$$

$$Det(L_s) = \frac{(1+\phi)(1-\alpha)}{\gamma\theta_N}k_f\left(1 - \rho_{\pi_f}\right) \tag{79}$$

where $\Omega = \frac{-(1+\phi)}{\theta_N\gamma r k_f}$. Once more it will become useful to remember that $Trace(L_s) = \omega_1 + \omega_2 + \omega_3$, $S_2(L_s) = \omega_1\omega_2 + \omega_1\omega_3 + +\omega_2\omega_3$ and $Det(L_s) = \omega_1\omega_2\omega_3$ where the $\omega_i$ 's correspond to the eigenvalues for $L_s$ (See theorem 1.2.12 from Horn and Johnson (1985)). For a) first notice that $\rho_{\pi_f} < 1 < \frac{1}{\alpha}$ then using expressions (77) and (79) we conclude that $Trace(L_s) > 0$ and $Det(L_s) < 0$. Using this result and Theorem 1.2.12 from Horn and Johnson (1985) we can infer that $L_s$ has one eigenvalue with a negative real part and two eigenvalues with positive real parts. Given that there are three jump variables ($\pi_f, \pi_N$, $c_N$), the number of jump variables is greater than the number of explosive roots. Applying the results of Blanchard and Kahn (1980) and Buiter (1984) it follows that there is real indeterminacy.

For b) and c) we divide the region in the positive plane $\rho_{\pi_f}$ vs $\alpha$ for which $1 < \rho_{\pi_f}$ and $0 < \alpha < 1$ in three exclusive subregions: subregion I for which $\rho_{\pi_f} > \frac{1}{\alpha}$, subregion II for which $\frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha} < \rho_{\pi_f} < \frac{1}{\alpha}$ and subregion III for which $1 < \rho_{\pi_f} < \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$. Remember that $0 < \Omega < 1$.

For subregion I note that if $\rho_{\pi_f} > \frac{1}{\alpha}$ then using expressions (78) and (79) we can deduce that $S_2(L_s) < 0$ and $Det(L_s) > 0$. For subregion II observe that since $0 < \Omega < 1$ and $0 < \alpha < 1$ then $1 < \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha} < \frac{1}{\alpha}$. Thus if $\frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha} < \rho_{\pi_f} < \frac{1}{\alpha}$ then using expressions (78) and (79) we can deduce that $S_2(L_s) < 0$ and $Det(L_s) > 0$. In both subregions I and II we obtain that $S_2(L_s) < 0$ and $Det(L_s) > 0$. This result and Theorem 1.2.12 from Horn and Johnson (1985) allow us to conclude that $L_s$ has one eigenvalue with a positive real part and two eigenvalues with negative real parts which in turn implies that there is real indeterminacy.

For subregion III, notice that if $1 < \rho_{\pi_f} < \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$ then using expressions (77), (78) and (79) we can conclude that $Trace(L_s) > 0$, $S_2(L_s) > 0$ and $Det(L_s) > 0$ which in turn implies that $L_s$ may have either three eigenvalues with positive parts or one eigenvalue with a positive real part and two eigenvalues with negative real parts. Hence there is either real determinacy or real indeterminacy.

To do a better characterization of the equilibrium when $1 < \rho_{\pi_f} < \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$ (subregion III) we need to apply the Theorem by Routh-Hurwicz[37]. As mentioned above this theorem states that the number of roots of $L_s$ with positive real parts is equal to the number of variations of sign in the scheme

$$1 \qquad -Trace(L_s) \qquad \frac{S_2(L_s)Trace(L_s) - Det(L_s)}{Trace(L_s)} \qquad -Det(L_s)$$

Hence, given that $Trace(L_s) > 0$ and $Det(L_s) > 0$ we need to find the sign of $\xi(\alpha, \rho_{\pi_f}) = S_2(L_s)Trace(L_s) - Det(L_s)$.

Recalling (77), (78) and (79) we can write

$$\xi(\alpha, \rho_{\pi_f}) = (1 - \alpha\rho_{\pi_f})(r + k_f(1 - \alpha\rho_{\pi_f})) - \Omega(1-\alpha)(r + (1-\alpha)\rho_{\pi_f} k_f) \tag{80}$$

and applying the Implicit Function Theorem and using (78) we can derive that

$$\frac{\partial \rho_{\pi_f}}{\partial \alpha} = \frac{-\frac{2}{r}\rho_{\pi_f} S_2(L_s) - (\rho_{\pi_f} - \Omega)}{2\alpha k_f(1 - \alpha\rho_{\pi_f}) + \alpha r + \Omega(1-\alpha)^2 k_f}$$

Given that $0 < \Omega < 1 < \rho_{\pi_f} < \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha} < \frac{1}{\alpha}$ and that $S_2(L_s) > 0$ then we deduce that $\frac{\partial \rho_{\pi_f}}{\partial \alpha} < 0$.

We can actually find the function that satisfies $\xi(\alpha, \rho_{\pi_f}) = 0$ solving the quadratic equation (80). Doing so we obtain the roots

$$\rho^L_{1,2} = \frac{1}{\alpha} + \frac{\Omega(1-\alpha)^2}{2\alpha^2} + \frac{r}{2\alpha k_f} \pm \frac{\sqrt{\zeta}}{2\alpha^2 k_f} \tag{81}$$

where $\zeta = (\Omega(1-\alpha)^2 k_f + \alpha r)^2 + 4\Omega k_f \alpha(1-\alpha)((1-\alpha)k_f + \alpha r)$.

We are not interested in $\rho^L_1 = \frac{1}{\alpha} + \frac{\Omega(1-\alpha)^2}{2\alpha^2} + \frac{r}{2\alpha k_f} + \frac{\sqrt{\zeta}}{2\alpha^2 k_f}$ since $\rho^L_1 > \frac{1}{\alpha}$. Therefore we concentrate on the root

$$\rho^L = \rho^L_2 = \frac{1}{\alpha} + \frac{\Omega(1-\alpha)^2}{2\alpha^2} + \frac{r}{2\alpha k_f} - \frac{\sqrt{\zeta}}{2\alpha^2 k_f} \tag{82}$$

[37] See Gantmacher (1960) for the Theorem of Routh-Hurwicz.

See Figure 6. There are several properties of $\rho^L$ that are important for our analysis. It is a real continuous function. It is straightforward to show that $\lim_{\alpha\to 1}\rho^L = 1$ and using L'Hopital rule that $\lim_{\alpha\to 0}\rho^L = \frac{1}{\Omega} + \frac{r(1-\Omega)}{\Omega k_f} > 1$. Moreover we can prove that $1 \leq \rho^L$ for any $0 < \alpha < 1$. The proof goes by contradiction. Assume that $\rho^L < 1$ then $\frac{1}{\alpha} + \frac{\Omega(1-\alpha)^2}{2\alpha^2} + \frac{r}{2\alpha k_f} - \frac{\sqrt{\zeta}}{2\alpha^2 k_f} < 1$ that in turn implies after some algebra manipulations that $0 < -4\alpha^2(1-\alpha)(1-\Omega)k_f((1-\alpha)k_f + r)$. However this is a contradiction since $0 < \Omega < 1$, $0 < \alpha < 1$, $k_f > 0$ and $r > 0$.

Similarly we can prove that $\rho^L \leq \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$. Once more the proof goes by contradiction. Suppose that $\frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha} < \rho^L$ then $\frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha} < \frac{1}{\alpha} + \frac{\Omega(1-\alpha)^2}{2\alpha^2} + \frac{r}{2\alpha k_f} - \frac{\sqrt{\zeta}}{2\alpha^2 k_f}$. After some algebra manipulations we obtain $0 < -k_f(1-\alpha)(1-\Omega)$. But this is a contradiction given that $0 < \Omega < 1$, $0 < \alpha < 1$ and $k_f > 0$.

Finally observe that when $\rho^L = \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$ then $\xi(\alpha, \rho_{\pi_f}) = S_2(L_s)Trace(L_s) - Det(L_s) < 0$. This result and $\rho^L \leq \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$ imply that if $\rho_{\pi_f} < \rho^L$ then $S_2(L_s)Trace(L_s) - Det(L_s) < 0$. On the other hand, if $\rho^L < \rho_{\pi_f}$ then $S_2(L_s)Trace(L_s) - Det(L_s) > 0$.

Therefore our best characterization of the equilibrium in the subregion III, that is when $1 < \rho_{\pi_f} < \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$, is the following . If $\rho^L < \rho_{\pi_f} < \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$ then $Trace(L_s) > 0$, $Det(L_s) > 0$ and $S_2(L_s)Trace(L_s) - Det(L_s) < 0$ which implies by the Theorem of Routh and Hurwicz that there is one eigenvalue with a positive real part and two eigenvalues with negative real parts. Hence since there are three jump variables $(\pi_f, \pi_N, c_N)$, the number of jump variables is greater than the number of explosive roots. Applying the results of Blanchard and Kahn (1980) and Buiter (1984) it follows that near the steady state there exists an infinite number of perfect foresight equilibria converging to the steady state. On the other hand if $1 < \rho_{\pi_f} < \rho^L$ then $Trace(L_s) > 0$, $Det(L_s) > 0$ and $S_2(L_s)Trace(L_s) - Det(L_s) > 0$ which implies by the Theorem of Routh and Hurwicz that there are three eigenvalue positive real part. Hence since there are three jump variables $(\pi_f, \pi_N, c_N)$, the number of jump variables is the same as the number of explosive roots. Applying the results of Blanchard and Kahn (1980) and Buiter (1984) it follows that there is real determinacy. ■

**Proof of Proposition 9**

**Proof.** For this proof it will be useful to study the sign of $S_2(L_s)$, $Trace(L_s)$ and $Det(L_s)$. Therefore their derived expressions in the proof for Proposition 8 will be used. Recalling (78) it is clear that the sign of $S_2(L_s)$ will be determined by the sign of $\left(\frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha} - \rho_{\pi_f}\right)$. In particular when $\rho_{\pi_f} = \rho^U = \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$ then $S_2(L_s) = 0$. Moreover if $\rho_{\pi_f} > \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$ then $S_2(L_s) < 0$ whereas if $\rho_{\pi_f} < \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$ then $S_2(L_s) > 0$. Furthermore it is important to observe some properties of the function $\rho^U = \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$. First note that $\frac{\partial \rho^U}{\partial \alpha} = -\frac{(1-\Omega)}{\alpha^2} > 0$ given that $\Omega > 1$. Second, notice that $\underset{\alpha\to 0}{lim}\ \rho^U = -\infty$ and $\underset{\alpha\to 1}{lim}\ \rho^U = 1$ and more importantly $\rho^U \leq 1$ for every $0 < \alpha < 1$.

In order to prove the proposition we consider two cases. For the first case $\rho_{\pi_f} < 1$ and therefore $\rho_{\pi_f} < \frac{1}{\alpha}$. In this case, using expressions (77) and (79) we can derive that $Trace(L_s) > 0$ and

$Det(L_s) < 0$. This result and Theorem 1.2.12 from Horn and Johnson (1985) allow us to conclude that in this case $L_s$ has one eigenvalue with a negative real part and two eigenvalues with positive real parts. For the second case we consider $\rho_{\pi_f} > 1$ and by the properties of the function $\rho^U = \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$ we also know that for this case $\rho_{\pi_f} > 1 \geq \rho^U$. Hence using expressions (78) and (79) we can conclude that $S_2(L_s) < 0$ and $Det(L_s) > 0$. This result and Theorem 1.2.12 from Horn and Johnson(1985) allow us to conclude that in this case $L_s$ has one eigenvalue with a positive real part and two eigenvalues with negative real parts.

Given that in both cases, $\rho_{\pi_f} < 1$ and $\rho_{\pi_f} > 1$, the number of explosive roots is smaller than the number of jump variables, $(\pi_f, \pi_N, c_N)$, we can apply the results of Blanchard and Kahn (1980) and Buiter (1984) to conclude that near the steady state there exists an infinite number of perfect foresight equilibria converging to the steady state. ■

**Proof of Proposition 10**

**Proof.** For this proof it is useful to derive expressions for the trace, the sum of the $2 \times 2$ principal minors and the determinant of the matrix $W_s$ in (65)

$$Trace(W_s) = k_p \alpha \left( \rho_{\pi_p} - \frac{1}{\alpha} + \frac{r}{\alpha k_p} \right) \tag{83}$$

$$S_2(W_s) = k_p r \alpha \left( \rho_{\pi_p} - \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha} \right) \tag{84}$$

$$Det(W_s) = \frac{(1+\phi)(1-\alpha)}{\gamma \theta_N} k_p \left( \rho_{\pi_p} - 1 \right) \tag{85}$$

where $\Omega = \frac{-(1+\phi)}{\theta_N \gamma r k_f}$. Once more it will become useful to remember that $Trace(W_s) = \omega_1 + \omega_2 + \omega_3$, $S_2(W_s) = \omega_1\omega_2 + \omega_1\omega_3 + +\omega_2\omega_3$ and $Det(W_s) = \omega_1\omega_2\omega_3$ where the $\omega_i$ 's correspond to the eigenvalues for $W_s$ (See theorem 1.2.12 from Horn and Johnson (1985)).

For a) observe that if $\rho_{\pi_p} < 1$ then using expressions (83) and (85) we can conclude that $S_2(W_s) < 0$ and $Det(W_s) > 0$. Using this result and Theorem 1.2.12 from Horn and Johnson (1985) we can infer that $W_s$ has one eigenvalue with a positive real part and two eigenvalues with negative real parts. Given that there are two jump variables $(\pi_N, c_N)$, the number of jump variables is greater than the number of explosive roots. Applying the results of Blanchard and Kahn (1980) and Buiter (1984) it follows it follows that near the steady state there exists an infinite number of perfect foresight equilibria converging to the steady state.

For b) we have to consider two cases: case 1 when $k^p < r$ and case 2 when and $k^p > r$.

For case 1, since $\rho_{\pi_p} > 1$ then recall (85) to derive that $Det(W_s) < 0$. Moreover since $\rho_{\pi_p} > 1$ and $k^p < r$ then rewriting (83) as $Trace(W_s) = k_p \alpha \left( \rho_{\pi_p} - \frac{1}{\alpha} \left( \frac{k_p - r}{k_p} \right) \right)$ we can conclude that $Trace(W_s) > 0$. Using this in tandem with $Det(W_s) < 0$ and Theorem 1.2.12 from Horn and Johnson (1985) we can infer that $W_s$ has one eigenvalue with a negative real part and two eigenvalues with positive real parts.

For case 2, that is, when $k^p > r$, we divide the region in the positive plane $\rho_{\pi_p}$ vs $\alpha$ for which $\rho_{\pi_p} > 1$ and $0 < \alpha < 1$ in two exclusive subregions: subregion we for which $1 < \rho_{\pi_p} < \frac{1}{\alpha} - \frac{r}{\alpha k_p}$ and subregion II for which $\frac{1}{\alpha} - \frac{r}{\alpha k_p} < \rho_{\pi_p}$ and $\rho_{\pi_p} > 1$. For subregion I note that if $1 < \rho_{\pi_p} < \frac{1}{\alpha} - \frac{r}{\alpha k_p}$ then using expressions (83) and (85) we can deduce that $Trace(W_s) < 0$ and $Det(W_s) < 0$. Moreover since $\frac{1}{\alpha} - \frac{r}{\alpha k_p} < \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$ for this subregion it is also true that $\rho_{\pi_p} < \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$ and therefore using (84) we conclude that $S_2(W_s) < 0$. Utilizing this in tandem with $Det(W_s) < 0$ and Theorem 1.2.12 from Horn and Johnson (1985) we can infer that $W_s$ has one eigenvalue with a negative real part and two eigenvalues with positive real parts.

For region II since $\frac{1}{\alpha} - \frac{r}{\alpha k_p} < \rho_{\pi_p}$ and $\rho_{\pi_p} > 1$ then using (83) and (85) we can conclude that $Trace(W_s) > 0$ and $Det(W_s) < 0$. This result together with Theorem 1.2.12 from Horn and Johnson (1985) imply that $W_s$ has one eigenvalue with a negative real part and two eigenvalues with positive real parts.

Summarizing we have just shown that when $\rho_{\pi_p} > 1$ then $W_s$ has one eigenvalue with a negative real part and two eigenvalues with positive real parts. Given that number of jump variables ($\pi_N$, $c_N$) is equal to the number of jump variables we can apply the results of Blanchard and Kahn (1980) and Buiter (1984) to state that there exists a unique perfect foresight equilibrium. ■

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