Board of Governors of the Federal Reserve System

International Finance Discussion Papers

Number 867R

April 2008

Closing large open economy models

Martin Bodenstein

Closing large open economy models*

Martin Bodenstein**
Federal Reserve Board

First version: August 2006
This version: March 2011

Abstract

A large class of international business cycle models admits multiple locally isolated deterministic steady states, if the elasticity of substitution between traded goods is sufficiently low. I explore the conditions under which such multiplicity occurs and characterize the dynamic properties in the neighborhood of each steady state. Models with standard incomplete markets, portfolio costs, a debt-elastic interest rate, or an overlapping generations framework allow for multiple steady states, if the model features multiple steady states under financial autarchy. If the excess demand for the foreign traded good is increasing in the good's own price in a given steady state, the equilibrium dynamics around this steady state are unbounded. Otherwise, the dynamics are bounded and unique. By contrast, with Uzawa-type preferences, the steady state is always unique and the associated equilibrium dynamics are always bounded and unique. The same results obtain under complete markets.

Keywords: stationarity, incomplete markets, open economy, multiple equilibria
JEL Classification: D51 F41

* I have benefited from conversations with Roc Armenter, Fabio Ghironi, Sylvain Leduc, and Thomas Lubik. I am particularly grateful to two anonymous referees and Giancarlo Corsetti (the editor) whose suggestions have substantially improved the paper. All remaining errors are mine. The views expressed in this paper are solely the responsibility of the author and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or any other person associated with the Federal Reserve System.
** Contact information: Martin Bodenstein: phone (202) 452 3796, Martin.R.Bodenstein@frb.gov.

1 Introduction

The class of international business cycle models nested in the framework of Corsetti, Dedola, and Leduc (2008) admits multiple steady states with zero net foreign asset holdings, if the elasticity of substitution between traded goods is sufficiently low. This paper explores the conditions under which such multiplicity occurs and characterizes the models' dynamic properties in the neighborhood of each steady state.

Equilibrium multiplicity is a pervasive feature of models with heterogenous agents. To build intuition consider the case of a static two-country endowment economy with two traded goods that are imperfect substitutes as in Kehoe (1991) and Mas-Colell, Whinston, and Green (1995). For simplicity, let the countries be mirror images of each other with respect to preferences and endowments. One equilibrium always features a relative price of the traded goods equal to unity. With home bias in consumption and a low elasticity of substitution between the traded goods, two more equilibria occur. If the price of the domestic good is high relative to the price of the foreign good, domestic agents are wealthy compared to foreign agents. Under a low elasticity of substitution, foreigners are willing to give up most of their good in order to consume at least some of the domestic good, and domestic agents end up consuming most of both goods. The reverse is true as well. Foreign agents consume most of the goods, if the foreign good is expensive in relative terms.

This intuition carries over to richer models of the international business cycle that feature international borrowing and lending, endogenous production, intertemporal savings and investment decisions, or non-traded goods. For these models to feature multiple steady states with zero net foreign assets under an otherwise standard calibration the elasticity of substitution between traded goods has to lie below 0.5.[1] Whereas in most studies the elasticity of substitution between traded goods and the trade (price) elasticity coincide, the two concepts differ in the model of Corsetti, Dedola, and Leduc (2008) due to the presence of non-traded distribution services. If the model is parameterized as in Corsetti, Dedola, and Leduc (2008),

[1] Prominent examples of such models are found in Backus, Kehoe, and Kydland (1995), Baxter and Crucini (1995), Heathcote and Perri (2002), and Stockman and Tesar (1995). However, none of these papers explores the possibility of multiple steady states, as the elasticity of substitution is generally assumed to be above unity.

multiple steady states occur for an elasticity of substitution between traded goods around unity, although the implied trade elasticity lies in the neighborhood of 0.5.

For a symmetric parameterization of the model I generally find three steady states. However, the model by Corsetti, Dedola, and Leduc (2008) can be shown to have at least five steady states for some parameterizations.[2]

The multiplicity of the steady state price vector occurs whether international financial markets are absent from the model or one focuses on an incomplete financial markets framework with zero net foreign assets in steady state.[3] This problem is unrelated to the issues about incomplete markets models addressed in Schmitt-Grohé and Uribe (2003) and many others. In standard incomplete markets model with one non-state-contingent bond the deterministic steady state of the net foreign asset position is not determined and the dynamics of the net foreign asset position as derived from a linear approximation of the model around a deterministic steady state are not stationary.[4] Absent arbitrage opportunities, the price of the non-state-contingent bond implies that expected marginal utility growth is equalized across countries. In the deterministic steady state, this condition contains no information about the steady state values of the system and the system of equilibrium conditions becomes underdetermined. In particular, any net foreign asset position is compatible with a steady state.

To determine the steady state position of net foreign assets and to remove the non-stationarity of its dynamics, I modify the original model similar to Schmitt-Grohé and Uribe (2003) by allowing for portfolio costs, a debt-elastic interest rate, or an endogenous discount factor. This list is augmented by the overlapping generations structure of Weil (1989) as implemented in Ghironi (2006) and Ghironi (2008). I show that the choice of a stationar-

[2] In the working paper version of Corsetti and Dedola (2005), the authors point out that for the case of an endowment economy a model with distribution costs admits multiple equilibria in the absence of international borrowing and lending. However, no systematic exploration of this feature is conducted.

[3] Multiplicity of the steady state price vector can also occur if the net foreign asset position differs from zero in steady state. However, the assumption of zero net foreign assets in steady state is widely made in the literature and implies that the steady states of the incomplete markets model coincide with those obtained in a model without financial markets.

[4] While throughout the analysis the only asset that trades internationally is one non-state-contingent bond, the issues raised carry over to environments with more assets such as those presented in Devereux and Sutherland (2008) as long as the available assets do not complete the market.

ity inducing device is not innocuous as it affects both the number of steady states in a low elasticity environment and the dynamics of the model around a steady state.

With portfolio costs, agents face a non-zero cost for bond holdings that differs from a reference level for international bond holdings specified exogenously by the researcher. Furthermore, the steady state is unique and stable only if the model with financial autarchy (or equivalently with incomplete markets and a zero net foreign asset position) has a unique steady state. If the original model has N steady states, the model with portfolio costs has N steady states. Those steady states for which the excess demand of the foreign good is decreasing in its relative price are associated with unique and locally bounded equilibrium dynamics. If the excess demand function is increasing in its relative price in a given steady state, the local equilibrium dynamics are not bounded. Interestingly, if multiple steady states occur under a symmetric calibration, it is typically the symmetric steady state that is associated with unbounded dynamics. Similar results are obtained for the cases of the debt-elastic interest rate and the overlapping generations framework.

Following Uzawa (1968), when the discount factor is assumed to be endogenous, an agent's rate of time preference is strictly decreasing in the agent's utility level. With strictly concave preferences and technologies the relative price of traded goods is uniquely pinned down under endogenous discounting given the function of the discount factor. The net foreign asset position is merely a residual. The equilibrium dynamics in the neighborhood of the unique steady state are always unique and locally bounded irrespective of the number of steady states in the original model with incomplete markets.[5]

Schmitt-Grohé and Uribe (2003) also analyze the case of complete markets. In this case, the net foreign asset position is a residual that does not enter the equilibrium dynamics as a state variable. The steady state of such a model is always unique and the associated equilibrium dynamics are unique and bounded.

Both Schmitt-Grohé and Uribe (2003) and Kim and Kose (2003) find that for the case of a small open economy the various approaches imply virtually identical dynamics. However,

[5]If the discount factor was increasing in the agent's utility level, the dynamics around any steady state would always be unbounded.

generalizing this finding to richer models as made by many researchers, may not be appropriate. Boileau and Normandin (2008) extend the analysis in Schmitt-Grohé and Uribe (2003) to a two-country model with one homogeneous good. Quantitative differences can occur in their setup depending on the persistence of technology shocks.

This paper exclusively analyzes the local dynamics around a given steady state. However, in the presence of multiple steady states, global solution techniques may find richer dynamics than local solution techniques. Bodenstein (2010) presents an analysis of the global dynamics in the model of Backus, Kehoe, and Kydland (1995) under endogenous discounting when the elasticity of substitution between traded goods is sufficiently low.

Remains to address the empirical relevance of models with low trade and substitution elasticities. For aggregate data Whalley (1984) reports a trade elasticity of 1.5. Hooper, Johnson, and Marquez (1998) report a short-run trade elasticity of 0.6 for the U.S. and values between 0 and 0.6 for the remaining G7 countries, while Taylor (1993) finds a short-run trade elasticity of 0.22. Using lower levels of aggregation, Broda and Weinstein (2006) report mean estimates for the elasticity of substitution for various pairs of traded goods between 4 and 6.

Applied macroeconomic studies, have commonly parameterized the substitution elasticity between traded goods at a value between 1 and 1.5 (see e.g. Backus, Kehoe, and Kydland (1995), Chari, Kehoe, and McGrattan (2002), and Heathcote and Perri (2002)). However, in line with the macroeconometric evidence, various authors have recently argued in favor of low values of the trade elasticity which coincides with the elasticity of substitution between traded goods for these studies. Heathcote and Perri (2002), Benigno and Thoenissen (2008) and Collard and Dellas (2007) show improved model performance with respect to key features of the international business cycle when allowing for substitution elasticities below 0.5.

Burstein, Neves, and Rebelo (2003), Corsetti and Dedola (2005), and Corsetti, Dedola, and Leduc (2008) refrain from assuming such low substitution elasticities directly, but instead introduce distribution costs in terms of non-traded goods to obtain a low implied value of the trade elasticity despite allowing the elasticity of substitution between traded goods to be around unity. The model in Corsetti, Dedola, and Leduc (2008) successfully addresses two important puzzles in international economics: the high volatility of the real exchange rate relative to fundamentals and the observed negative correlation between the real exchange rate and relative consumption (Backus and Smith (1993)).

As Kollmann (2006) shows, a low elasticity of substitution between traded goods may also be responsible for the apparent home bias in equity holdings. Rabanal and Tuesta (2010) estimate a DSGE model with sticky prices using a Bayesian approach. Their median estimates for the elasticity of substitution range from 0.01 to 0.91 for different specifications of their model. Lubik and Schorfheide (2006) estimate the elasticity of substitution to be around 0.4.

The remainder of the paper is structured as follows. Section 2 introduces the issues considered in this paper in a simple model. Section 3 lays out the general model, which is parameterized in Section 4. Steady state multiplicity is discussed in Section 5, while the local dynamics of the model are discussed in Section 6. Concluding remarks are offered in Section 7. The paper is accompanied by a separate Technical Appendix.

2 Simple example

Consider a two-country, two-good endowment economy with incomplete international financial markets. The two countries are each inhabited by a continuum of identical agents of measure 1. This model extends the introductory example in Corsetti, Dedola, and Leduc (2008) to a dynamic environment. I first illustrate under what restrictions on the value of the elasticity of substitution between traded goods the model admits multiple deterministic steady states. Second, I relate steady state multiplicity to the question when the local dynamics around a given steady state are locally bounded or unbounded.

Each period $t+j$, agents in country i obtain $y^T_{i,t+j}$ units of the traded good i. Agents consume both the home and the foreign good and maximize their expected discounted lifetime utility. The only asset that trades internationally is one non-state-contingent bond. More formally, a representative agent faces:

$$\max_{\substack{c_{i1,t+j}, c_{i2,t+j}, \\ b_{i,t+j}}} \widetilde{\mathbb{E}}_t \left\{ \sum_{j=0}^{\infty} \beta^j ln \left(c_{i,t+j}^T \right) \right\} \tag{1}$$

s.t.

$$P_{1,t+j} c_{i1,t+j} + P_{2,t+j} c_{i2,t+j} \leq P_{i,t+j} y_{i,t+j}^T + b_{i,t-1+j} - Q_{i,t+j} b_{i,t+j}. \tag{2}$$

All variables are expressed in per capita terms. c_{im} is the consumption of good m by agents of country i. P_i is the price of good i. b_i denotes holdings of a non-state-contingent bond with price Q_i. To rule out Ponzi-schemes, I assume that agents face an upper bound for borrowing $\tilde{b}_i$ that is large enough to never bind in this application. The aggregate consumption good c_i^T satisfies:

$$c_{i,t+j}^T = \left[\left(\alpha_{i1}^T \right)^{\frac{1}{\varepsilon^T}} \left(c_{i1,t+j} \right)^{\frac{\varepsilon^T - 1}{\varepsilon^T}} + \left(\alpha_{i2}^T \right)^{\frac{1}{\varepsilon^T}} \left(c_{i2,t+j} \right)^{\frac{\varepsilon^T - 1}{\varepsilon^T}} \right]^{\frac{\varepsilon^T}{\varepsilon^T - 1}}, \tag{3}$$

with $0 < \alpha_{im}^T < 1$ for $i = 1, 2$ and $m = 1, 2$. The elasticity of substitution between the two traded goods is denoted by ε^T. Market clearing requires:

$$c_{11,t+j} + c_{21,t+j} = y_{1,t+j}^T, \tag{4}$$

$$c_{12,t+j} + c_{22,t+j} = y_{2,t+j}^T, \tag{5}$$

$$b_{1,t+j} + b_{2,t+j} = 0. \tag{6}$$

The price of the aggregate consumption good in country 1 is taken to be the numéraire.

2.1 Steady state multiplicity

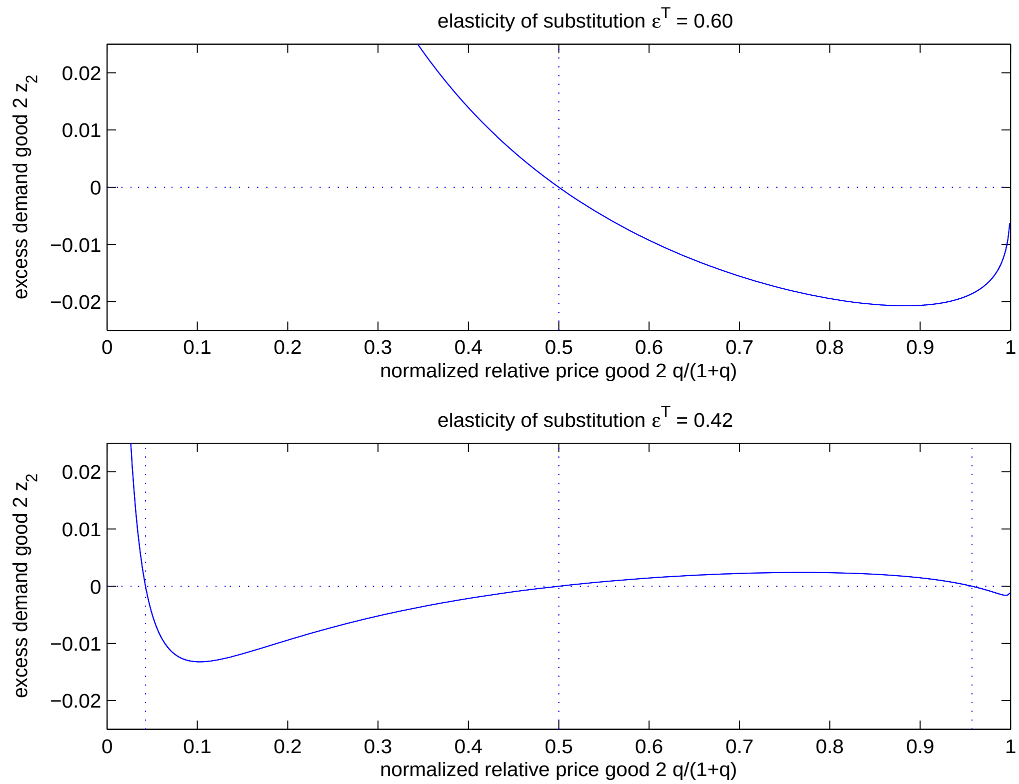
Under the assumption that the net foreign asset position is zero in steady state ($b_i = 0$), this model can display multiple distinct steady states if the elasticity of substitution between the two traded goods is sufficiently low. The relative price $q = \frac{P_2}{P_1}$ constitutes a steady state equilibrium, if the excess demand function for good 2 satisfies:

$$z_2(q) \equiv c_{12}(q) + c_{22}(q) - y_2^T(q)$$

$$z_2(q) \leq 0,\ \infty \geq q \geq 0 \text{ and } q z_2(q) = 0. \tag{7}$$

Applying standard fixed-point theorems, it can be shown that a steady state exists for this model. Furthermore, by virtue of the index theorem, see Kehoe (1991) and Mas-Colell, Whinston, and Green (1995), the number of steady state equilibria needs to be odd. Furthermore, if the slope of the excess demand function is positive for a given steady state value of q with zero net foreign assets, there must be at least two more values of q for which $z_2(q) = 0$.

Figure 1: Excess demand function and steady state multiplicity



Notes: The figure plots the excess demand function for good 2 as a function of the normalized relative price $q/(1+q)$ with $\alpha_{11}^T = \alpha_{22}^T = 0.9$, $\alpha_{im}^T = 1 - \alpha_{ii}^T$, and $y_1^T = y_2^T = 1$. The solid line depicts the excess demand function. The dotted vertical lines indicate the zeros of the excess demand function. For $\varepsilon^T = 0.6$, the unique zero features $q/(1+q) = 0.500$. For $\varepsilon^T = 0.42$, the three zeros occur at $q/(1+q)$ equal to 0.043, 0.500, and 0.957, respectively.

To advance the analysis, I set $\alpha_{ii}^T = \alpha_{mm}^T$, $\alpha_{ii}^T = 1 - \alpha_{im}^T$, for $i = 1, 2$ and $m = 1, 2$ and fix $y_1^T = y_2^T$ in steady state. Under these assumptions, one steady state always features $q = 1$, i.e., $\frac{q}{1+q} = 0.5$, irrespective of the value of ε^T. Figure 1 plots the excess demand for good 2 as a function the normalized price $\frac{q}{1+q}$ for a high and a low value of ε^T. The remaining parameter values are $\alpha_{11}^T = \alpha_{22}^T = 0.9$, $\alpha_{im}^T = 1 - \alpha_{ii}^T$, and $y_1^T = y_2^T = 1$. In the case of $\varepsilon^T = 0.6$, the excess demand function equals zero only for $q = 1$. Furthermore, the slope of the excess demand function is negative at $q = 1$. For the lower value of ε^T of 0.42, the slope of the excess demand function is positive at the steady state equilibrium with $q = 1$. Consequently, there are two more price equilibria: one for $\frac{q}{1+q} = 0.043$ and the other one for $\frac{q}{1+q} = 0.957$.

As shown in Appendix A, the slope of the excess demand function in a deterministic steady state is given by:

$$\frac{\partial z_2}{\partial q} q \quad = \quad c_{12}(q) \left\{ \left(1 - \varepsilon^T\right) \left(\frac{c_{11}(q)}{y_1^T} + \frac{c_{22}(q)}{y_2^T} \right) - 1 \right\}.$$

Balanced trade in steady state implies $\frac{c_{ii}(q)}{y_i^T} = \alpha_{ii}^T$ and the slope of the excess demand function for good 2 is then positive at $q = 1$, if:

$$\varepsilon^T \leq 1 - \frac{1}{2\alpha_{11}^T}. \tag{8}$$

Thus, the model features multiple steady states, whenever condition (8) is satisfied. For the parameterization underlying Figure 1 the threshold value for the elasticity of substitution at which steady state multiplicity occurs is 0.4444. The associated change in the sign of the slope of the excess demand function from negative to positive at a low value of the substitution elasticity lies behind what Corsetti, Dedola, and Leduc (2008) term “the negative transmission mechanism” and allows their model to be a candidate solution to the Backus-Smith puzzle.[6]

2.2 Local equilibrium dynamics

The incomplete markets model outlined in the previous section features two well-known problems if the decision rules are derived from a linear approximation around a deterministic steady state. First, any value of the net foreign asset position is compatible with a steady state. Second, the model is not stationary as the net foreign asset position follows a unit root process.

[6]Corsetti, Dedola, and Leduc (2008) argue that a positive supply shock to the home country can cause an appreciation of the home country’s terms of trade when the excess demand function is upward sloping in the steady state around which the model is linearized. Absent “the negative transmission mechanism” their model can resolve the Backus-Smith puzzle only for very persistent shock processes.

The first order conditions with respect to bond holdings imply, that expected marginal utility growth expressed in a common good needs to be equalized across countries:

$$\mathbb{E}_{t+j}\left[\beta\left(\frac{c^T_{1,t+1+j}}{c^T_{1,t+j}}\right)^{-1}\right] = \mathbb{E}_{t+j}\left[\beta\left(\frac{c^T_{2,t+1+j}}{c^T_{2,t+j}}\right)^{-1}\frac{rer_{t+j}}{rer_{t+1+j}}\right]. \tag{9}$$

In steady state, equation (9) reduces to an identity, leading to a system of N unknown endogenous variables, but $N-1$ equations. It is common practice to exogenously specify the steady state value of the net foreign asset position. For the remainder of the paper, I will assume that the net foreign position is zero in steady state. Thus, the model with incomplete markets has the same steady states as a model without internationally traded assets.

However, imposing the steady state level of net foreign assets exogenously does not remove the unit root of the net foreign asset position in the approximate model. Following among others Schmitt-Grohé and Uribe (2003), I introduce portfolio costs to render the net foreign asset position stationary.[7] Households incur the cost $\frac{1}{2}\gamma\left(\frac{b_{i,t+j}}{P_{i,t+j}}\right)^2 P_{i,t+j}$ for holding or issuing international debt. These costs are rebated lump-sum to the households. As portfolio costs are zero whenever the net foreign asset position is zero, the steady states in the model with portfolio costs are the same as in the model described in the previous subsection.

To study the dynamics around a given steady state, Appendix A shows that the unique state variable $\Delta b_{1,t-1+j}$ satisfies the system:[8]

$$\begin{pmatrix} -\frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial q}q}\beta & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Delta b_{1,t+1+j} \\ \Delta b_{1,t+j} \end{pmatrix} + \begin{pmatrix} \left(1+\beta+\beta\frac{\frac{\partial z_2}{\partial q}q}{\overline{dz_2}}\tilde{\gamma}\right)\frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial q}q} & -\frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial q}q} \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \Delta b_{1,t+j} \\ \Delta b_{1,t-1+j} \end{pmatrix} = 0. \tag{10}$$

$\frac{\partial z_2}{\partial q}q$ represents the slope of the excess demand function for good 2. The term $\overline{dz_2}$ is always negative in steady state. and the sign of $\frac{\partial z_2}{\partial dW_1}$ is immaterial for the subsequent analysis. The

[7] The net foreign asset position becomes non-stationary as a result of using local solution techniques. While higher order perturbation methods imply the same problems as a linear solution approach, global solution methods do not. If the model was solved using global methods, stationarity would be preserved due to the presence of the borrowing constraints $\tilde{b}_i$. In addition, the concept of a deterministic steady state would need to be replaced by that of a stationary distribution.

[8] $\Delta b_{i,t+j}$ is the absolute deviation of real bond holdings $b_{i,t+j}/P_{i,t+j}$ from zero.

portfolio costs imposed on the agents are represented by the parameter:

$$\check{\gamma} = \frac{\gamma}{\beta^2 \Phi_1(q)} \left[1 + \frac{1}{q} \right] > 0. \tag{11}$$

The term $\frac{\frac{\partial z_2}{\partial q} q}{dz_2} \check{\gamma}$ is positive (negative), whenever $\frac{\partial z_2}{\partial q} q$ is negative (positive) in the deterministic steady state around which the model is approximated.

The following theorem shows that steady states for which the excess demand function is upward-sloping display unbounded dynamics under the portfolio cost approach.

Theorem 1 *Around a given steady state, the eigenvalues of the dynamic system associated with the portfolio cost model λ_1 and λ_2 satisfy:*

1. *For $\gamma = 0$, $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{\beta}$. Thus, bond holdings follow a unit-root process (restating the non-stationarity problem under incomplete markets).*
2. *For $\gamma > 0$, and $\frac{\partial z_2}{\partial q} q < 0$ (a downward sloping excess demand function), $\lambda_1 < 1$ and $\lambda_2 > \frac{1}{\beta}$. Thus, the dynamics around the point of approximation are locally unique and bounded.*
3. *For $\gamma > 0$, and $\frac{\partial z_2}{\partial q} q > 0$ (an upward sloping excess demand function), $\lambda_1 > 1$ and $1 < \lambda_2 < \frac{1}{\beta}$. Thus, the dynamics around the point of approximation are not bounded.*

Proof. The proof follows directly from the characteristic equation associated with the approximate system:

$$\lambda^2 - \lambda \left(\frac{1+\beta}{\beta} + \frac{\frac{\partial z_2}{\partial q} q}{dz_2} \check{\gamma} \right) + \frac{1}{\beta} = 0. \tag{12}$$

If the number of eigenvalues larger than one in absolute value exceeds (falls short of) the number of state variables, the equilibrium dynamics are not bounded (indeterminate). If the number of such eigenvalues coincides with the number of state variables, the dynamics are unique and bounded. ■

An alternative approach of rendering the net foreign asset position stationary is due to Uzawa (1968). As shown in Section 5, the steady state under endogenous discounting is always unique for a given parameterization of the model. Replacing β^j in equation (1) by $\theta_{i,t+1+j} = \beta_i(c^T_{i,t+j})\theta_{i,t+j}$ with $\beta_i' < 0$ and, to ease exposition, assuming that agents do not internalize the effects of their current choices on future discount factors, one obtains:

Theorem 2 *Around the unique steady state, the eigenvalues of the dynamic system with an endogenous discount factor satisfy* $\lambda_1 < 1$ *and* $\lambda_2 = \frac{1}{\beta}$. *Thus, the equilibrium dynamics around a steady state are always locally unique and bounded irrespective of the slope of the excess demand function in this steady state.*

Proof. The proof follows directly from the characteristic equation associated with the approximate system:

$$\lambda^2 - \left(\frac{1+\beta}{\beta} + \frac{\widetilde{dz_2}}{\overline{dz_2}} \right) \lambda + \left(1 + \frac{\widetilde{dz_2}}{\overline{dz_2}} \right) \frac{1}{\beta} = 0, \tag{13}$$

where the additional term $\widetilde{dz_2}$ is always positive. ■

2.3 Intuition

Multiplicity of steady states In the model with portfolio costs, the steady state interest rate equals $\frac{1}{\beta}$. Hence, no country has an incentive to borrow or lend in steady state. All equilibria of the financial autarchy case are therefore valid steady states since they are compatible with $b_1 = 0$.

The model of endogenous discounting, however, dictates that for a given functional choice of the discount factor $\beta_1(q) = \beta_2(q)$ in the steady state. Uniqueness of the steady state price q^*_{endog} follows promptly: suppose that another price vector q^* that constitutes a steady state with $b_1 = b_2 = 0$ is also a steady state of the model with endogenous discounting. Let $q < q^*_{endog}$, which would imply that overall consumption in country 1 exceeds overall consumption in country 2, i.e., $c_1^T(q^*) > c_2^T(q^*)$. Thus, country 1 agents are willing to borrow resources at an interest rate of $\frac{1}{\beta_1(q^*)}$ while country 2 agents only demand $\frac{1}{\beta_2(q^*)} < \frac{1}{\beta_1(q^*)}$. Hence, country 1 finds it optimal to borrow from country 2 violating $b_1 = 0$.

Stability of steady states The logic behind the stability of the unique steady state in the model with endogenous discounting is closely related to the argument about its uniqueness. Assume that q is below its steady state value. This implies that consumption in country 1 (2) is above (below) its steady state value. Suppose, that the relative price is even lower in the next period, suggesting that the economy moves away from the steady state. This implies an increasing (decreasing) consumption profile in country 1 (2). In addition, the discount factor in country 1 (2) falls (rises). Hence, the price of the non-state-contingent bond falls in country 1 but rises in country 2. Obviously, the opposite movement of bond prices is inconsistent with the absence of arbitrage dictated by the risk sharing condition. Hence, if q is below its steady state value at time t, q must rise in $t+1$ and the economy converges to its unique steady state.

Consider the case of a steady state with $\frac{\partial z_2}{\partial q} q > 0$ in the bond economy with portfolio costs. The price of bonds consists of two pieces: the intertemporal marginal rate of substitution and the derivative of the portfolio costs. First, if q is slightly below its steady state value, overall consumption in country 1 (2) is above (below) its corresponding steady state value. Stability of the steady state requires q to rise and c_1^T to fall over time. As a result, the intertemporal marginal rate of substitution in country 1 (2) rises (falls), which leads to a divergence of bond prices. Second, when q rises, bond holdings and the derivative of the portfolio costs fall. This effect on bond prices is negative in both countries, but it is stronger in country 2 since portfolio costs are measured in terms of each country's good. This second effect operates towards a rise of the bond price in country 2 relative to country 1. However, the change in bond holdings is small owing to the fact that the excess demand function is fairly flat around such a steady state. Hence, bond prices would drift apart if a steady state with $\operatorname{sign}\left(\frac{\partial z_2}{\partial q} q\right) > 0$ was stable. Hence, such a steady must be associated by unbounded dynamics.

Using the model by Corsetti, Dedola, and Leduc (2008), the remainder of this paper explores the conditions for steady state multiplicity and the local dynamics under different stationarity inducing devices.

3 General model

Each of the two countries ($i = 1, 2$) produces two goods. The first good is traded internationally without trade frictions (traded good), while the other good is not traded (non-traded good). The home and foreign traded goods are imperfect substitutes in a household's utility function. The traded and the non-traded goods are produced by perfectly competitive firms and production requires the use of labor and capital. As an important deviation from earlier work, Corsetti, Dedola, and Leduc (2008) assume that consumption of traded goods requires non-traded distribution services as input.

3.1 Standard incomplete markets economy

The subsequent analysis explores the implications of different assumptions about international financial markets in the presence of multiple steady states. In laying out the full model, I assume that international financial markets are incomplete in the sense that the only asset that trades internationally is one non-state-contingent bond.

3.1.1 Households

The representative household in country i maximizes its expected discounted lifetime utility subject to its budget constraint. All variables are expressed in per household units:

$$\max_{\substack{c_{i,t+j},l_{i,t+j}\\ k_{i,t+j},x_{i,t+j},b_{i,t+j}}} \widetilde{\mathbb{E}}_t \left\{ \sum_{j=0}^{\infty} \beta^j U_i \left(c_{i,t+j}, l_{i,t+j} \right) \right\} \tag{14}$$

s.t.

$$P^C_{i,t+j} c_{i,t+j} + P^I_{i,t+j} x_{i,t+j} \leq P_{i,t+j} w_{i,t+j} l_{i,t+j} + P_{i,t+j} r_{i,t+j} k_{i,t-1+j}$$

$$+ P_{i,t+j} v_{i,t+j} + b_{i,t-1+j} - Q_{i,t+j} b_{i,t+j}, \tag{15}$$

$$k_{i,t+j} \leq (1 - \delta_i) k_{i,t-1+j} + x_{i,t+j}, \tag{16}$$

where c_i, x_i, l_i and k_i denote final consumption, final investment, labor, and capital, respectively. P_i^C is the price of the final consumption good, P_i^I is the price of the final investment good, P_i is the price of the traded good produced by country i, $P_i r_i$ and $P_i w_i$ are the nominal rental rate of capital and the nominal wage. Assuming local ownership of firms, $P_i v_i$ are the profits of the traded and non-traded goods producers that are located in country i. b_i denotes holdings of the non-state-contingent bond. Q_i is the price of this bond. To rule out Ponzi-schemes, I assume that agents face an upper bound for borrowing $\tilde{b}_i$ that is large enough to never bind in this application.

3.1.2 Firms

Both the producers of traded and non-traded goods utilize labor and capital and act in perfectly competitive factor and product markets. Technologies are of the constant elasticity of substitution type. Since capital is owned by households and rented out to firms, the maximization problem of a traded goods producer from country i can be written as:

$$\max_{l_{i,t+j}^T, k_{i,t+j}^T} F_i^T \left(l_{i,t+j}^T, k_{i,t+j}^T \right) - w_{i,t+j} l_{i,t+j}^T - r_{i,t+j} k_{i,t+j}^T, \tag{17}$$

where the production technology satisfies:

$$F_i^T \left(l_{i,t+j}^T, k_{i,t+j}^T \right) = \left[\left(\omega_{li}^T \right)^{1-\kappa_i^T} \left(A_{i,t+j}^T l_{i,t+j}^T \right)^{\kappa_i^T} + \left(\omega_{ki}^T \right)^{1-\kappa_i^T} \left(k_{i,t+j}^T \right)^{\kappa_i^T} \right]^{\frac{1}{\kappa_i^T}} \tag{18}$$

if $\kappa_i^T < 1$, and

$$F_i^T \left(l_{i,t+j}^T, k_{i,t+j}^T \right) = \left(\frac{A_{i,t+j}^T l_{i,t+j}^T}{\omega_{li}^T} \right)^{\omega_{li}^T} \left(\frac{k_{i,t+j}^T}{\omega_{ki}^T} \right)^{\omega_{ki}^T} \tag{19}$$

if $\kappa_i^T = 0$,

with $\omega_{li}^T + \omega_{ki}^T = 1$. Let y_i^T denote the supply of the traded good. The problem of non-traded goods producers is analogous with variables and parameters carrying the superscript N rather than T. Because of the normalization choice for factor prices, the objective function of such firms is given by:

$$\frac{P_{i,t+j}^N}{P_{i,t+j}} F_i^N \left(l_{i,t+j}^N, k_{i,t+j}^N \right) - w_{i,t+j} l_{i,t+j}^N - r_{i,t+j} k_{i,t+j}^N. \tag{20}$$

3.1.3 Trade and aggregation

Households in country i consume the final consumption good c_i and purchase the final investment good x_i. Shipping of the traded goods is costless.

Consumption good A household in country i purchases c_{i1} units of the traded good of country 1, c_{i2} units of the traded good of country 2, and c_i^N units of the non-traded good.

The three types of goods are combined to form the final consumption good according to:

$$c^T_{i,t+j} = \left[\left(\alpha^T_{i1}\right)^{\frac{1}{\varepsilon^T_i}} \left(c_{i1,t+j}\right)^{\frac{\varepsilon^T_i - 1}{\varepsilon^T_i}} + \left(\alpha^T_{i2}\right)^{\frac{1}{\varepsilon^T_i}} \left(c_{i2,t+j}\right)^{\frac{\varepsilon^T_i - 1}{\varepsilon^T_i}} \right]^{\frac{\varepsilon^T_i}{\varepsilon^T_i - 1}}, \tag{21}$$

$$c_{i,t+j} = \left[\left(\alpha^N_{i1}\right)^{\frac{1}{\varepsilon^N_i}} \left(c^T_{i,t+j}\right)^{\frac{\varepsilon^N_i - 1}{\varepsilon^N_i}} + \left(\alpha^N_{i2}\right)^{\frac{1}{\varepsilon^N_i}} \left(c^N_{i,t+j}\right)^{\frac{\varepsilon^N_i - 1}{\varepsilon^N_i}} \right]^{\frac{\varepsilon^N_i}{\varepsilon^N_i - 1}}, \tag{22}$$

with $0 < \alpha^T_{im} < 1$ and $0 < \alpha^N_{im} < 1$ for $i = 1, 2$, $m = 1, 2$. These quasi shares in the aggregators are assumed to sum up to 1. ε^N_i is the substitution elasticity between the traded goods aggregate and the non-traded good in country i. ε^T_i is the substitution elasticity between the home and foreign traded goods. As in Burstein, Neves, and Rebelo (2003) and Corsetti, Dedola, and Leduc (2008), the consumption of traded goods requires distribution services. Consuming one unit of any traded good goes along with a cost of η_i units of the country's non-traded good. Thus, minimizing the costs for c_i units of the final consumption good requires solving the problem:

$$\min_{\substack{c_{i1,t+j}, c_{i2,t+j}, \\ c^T_{i,t+j}, c^N_{i,t+j}}} \left(P_{1,t+j} + \eta_i P^N_{i,t+j}\right) c_{i1,t+j} + \left(P_{2,t+j} + \eta_i P^N_{i,t+j}\right) c_{i2,t+j} + P^N_{i,t+j} c^N_{i,t+j} \tag{23}$$

subject to (21) and (22).

Investment good The final investment good is an aggregate of traded goods only. In combining the two traded goods, the household solves:

$$\min_{x_{i1,t+j}, x_{i2,t+j}} P_{1,t+j} x^T_{i1,t+j} + P_{2,t+j} x^T_{i2,t+j} \tag{24}$$

s.t.

$$x_{i,t+j} = \left[\left(\alpha^I_{i1}\right)^{\frac{1}{\varepsilon^I_i}} \left(x_{i1,t+j}\right)^{\frac{\varepsilon^I_i - 1}{\varepsilon^I_i}} + \left(\alpha^I_{i2}\right)^{\frac{1}{\varepsilon^I_i}} \left(x_{i2,t+j}\right)^{\frac{\varepsilon^I_i - 1}{\varepsilon^I_i}} \right]^{\frac{\varepsilon^I_i}{\varepsilon^I_i - 1}}, \tag{25}$$

with $0 < \alpha^I_{im} < 1$ for $i = 1, 2$, $m = 1, 2$ and $\alpha^I_{i1} + \alpha^I_{i2} = 1$. The substitution elasticity between traded goods for investment is denoted by ε^I_i.

Relative prices For later reference, define the following relative prices: the relative price of country 2's good to country 1's good $q_{t+j} \equiv \frac{P_{2,t+j}}{P_{1,t+j}}$, the relative price of the non-traded good $q^N_{i,t+j} \equiv \frac{P^N_{i,t+j}}{P_{i,t+j}}$, the inverse relative price of final consumption $\Phi_{i,t+j} \equiv \frac{P_{i,t+j}}{P^C_{i,t+j}}$, the inverse relative price of final investment $\Pi_{i,t+j} \equiv \frac{P_{i,t+j}}{P^I_{i,t+j}}$. Thus, the consumption real exchange rate is given by $rer_{t+j} = \frac{\Phi_{2,t+j}}{\Phi_{1,t+j}} q_{t+j}$. In solving the model, aggregate consumption in country 1 is set to be the numéraire, i.e., $P^C_{1,t+j} = 1$.

3.1.4 Market clearing and equilibrium

Goods market clearing requires:

$$
\begin{aligned}
y^T_{i,t+j} &= \sum_{l=1}^{2} \left(c_{li,t+j} + x_{li,t+j} \right) \frac{\nu_l}{\nu_i}, && (26) \\
y^N_{i,t+j} &= c^N_{i,t+j} + \eta_i \left(c_{i1,t+j} + c_{i2,t+j} \right), && (27)
\end{aligned}
$$

for $i = 1, 2$ at all $t + j$. With $\nu_1 = 1$, differences in relative population size are captured by ν_2. Factor market clearing requires:

$$
\begin{aligned}
l^T_{i,t+j} + l^N_{i,t+j} &= l_{i,t+j}, && (28) \\
k^T_{i,t+j} + k^N_{i,t+j} &= k_{i,t-1+j}, && (29)
\end{aligned}
$$

for $i = 1, 2$ at all j. The bond market clears if:

$$
\sum_{i=1}^{2} b_{i,t+j} \nu_i = 0. \qquad (30)
$$

A competitive equilibrium is consequently defined as:

Definition 1 *A competitive equilibrium is a collection of allocations $c_{i1,t+j}$, $c_{i2,t+j}$, $c^T_{i,t+j}$, $c^N_{i,t+j}$, $c_{i,t+j}$, $x_{i1,t+j}$, $x_{i2,t+j}$, $x_{i,t+j}$, $k_{i,t-1+j}$, $l_{i,t+j}$, $y^T_{i,t+j}$, $k^T_{i,t+j}$, $l^T_{i,t+j}$, $y^N_{i,t+j}$, $k^N_{i,t+j}$, $l^N_{i,t+j}$, $b_{i,t+j}$, prices q_{t+j}, $q^N_{i,t+j}$, rer_{t+j}, $w_{i,t+j}$, $r_{i,t+j}$, $Q_{i,t+j}$, $\Phi_{i,t+j}$, $\Pi_{i,t+j}$ and profits $v_{i,t+j}$, for $i = 1, 2$ and all j, such that (i) for every household the allocations solve a household's maximization problem for given prices, (ii) for every firm profits are maximized, and (iii) the markets for labor, capital, goods, and bonds clear.*

3.1.5 Trade elasticity

Absent distribution services ($\eta_i = 0$ for $i = 1, 2$), the elasticity of substitution between traded goods coincides with the trade (price) elasticity. However, this is not the case for $\eta_i > 0$; the price of the home good relative to the foreign good at the consumer level no longer matches the terms of trade, i.e., the relative price between traded goods. The first order condition of the consumer problem implies:

$$c_{12,t+j} = \frac{\alpha_{12}^T}{\alpha_{11}^T} \left(\frac{1 + \eta_1 q_{1,t+j}^N}{q_{t+j} + \eta_1 q_{1,t+j}^N} \right)^{\varepsilon_1^T} c_{11,t+j}, \tag{31}$$

or after linearizing around a deterministic steady state:

$$\hat{c}_{11,t+j} - \hat{c}_{12,t+j} = \frac{\varepsilon_1^T}{1 + \eta_1 \frac{q_1^N}{q}} \hat{q}_{t+j} - \left[\frac{1}{1 + \eta_1 q_1^N} - \frac{1}{q + \eta_1 q_1^N} \right] \varepsilon_1^T \eta_1 q_1^N \hat{q}_{1,t+j}^N, \tag{32}$$

where $\hat{x}$ denotes the log-linear deviation of variable x from its steady state value.

Corsetti, Dedola, and Leduc (2008) refer to $\frac{\varepsilon_1^T}{1+\eta_1 \frac{q_1^N}{q}}$ as the trade elasticity. In addition to the elasticity of substitution between traded goods ε_1^T, the trade elasticity depends on the endogenous steady state values of both the relative price of traded goods q and the relative price of non-traded goods q_1^N.

3.2 Assumptions on international financial markets

I investigate the following deviations from the standard incomplete markets model that render the net foreign asset position stationary and determine its steady state:

1. Agents face portfolio costs for holding/issuing bonds as in Heathcote and Perri (2002) and Schmitt-Grohé and Uribe (2003). The budget constraint (15) changes to:

$$\begin{aligned} & P_{i,t+j}^C c_{i,t+j} + P_{i,t+j}^I x_{i,t+j} \leq P_{i,t+j} w_{i,t+j} l_{i,t+j} + P_{i,t+j} r_{i,t+j} k_{i,t-1+j} \\ & + P_{i,t+j} v_{i,t+j} + b_{i,t-1+j} - Q_{i,t+j} b_{i,t+j} - P_{i,t+j} \Gamma \left(b_{i,t+j} / P_{i,t+j} \right) + T_{i,t+j}. \end{aligned} \tag{33}$$

   The portfolio cost function Γ satisfies $\Gamma(0) = 0$, $\Gamma'(0) = 0$, and $\Gamma' > 0$ for $b_i > 0$ and $\Gamma' < 0$ for $b_i < 0$, and $\Gamma''(0) > 0$. $T_{i,t+j}$ is a lump-sum transfer in the amount of the collected fees. Consequently, condition (9) changes to:

$$\begin{aligned} & \mathbb{E}_{t+j} \left[\beta \frac{U_c(c_{1,t+1+j}, l_{1,t+1+j})}{U_c(c_{1,t+j}, l_{1,t+j})} \right] = \mathbb{E}_{t+j} \left[\beta \frac{U_c(c_{2,t+1+j}, l_{2,t+1+j})}{U_c(c_{2,t+j}, l_{2,t+j})} \frac{rer_{t+j}}{rer_{t+1+j}} \right] \\ & + \Gamma' \left(\frac{b_{1,t+j}}{P_{1,t+j}} \right) - \Gamma' \left(\frac{b_{2,t+j}}{P_{2,t+j}} \right). \end{aligned} \tag{34}$$

2. The interest rate is debt-elastic as in Boileau and Normandin (2008), Devereux and Smith (2007), and Schmitt-Grohé and Uribe (2003). The interest rate differential between the two countries is given by:

$$R_{1,t+j} = R_{2,t+j} \Psi \left(\frac{b_{1,t+1+j}}{P_{1,t+j}} - \bar{b}_1 \right), \tag{35}$$

with $R_{i,t+j} = 1/Q_{i,t+j}$. The function Ψ satisfies $\Psi(0) = 1$ and $\Psi' < 0$. $\bar{b}_1$ is a reference level of debt for country 1, which is set to zero. When country 1 is a net borrower, it faces an interest rate that is higher than the interest rate in country 2. When country 1 is a lender, it receives an interest rate that is lower. Condition (9) changes to:

$$\begin{aligned} &\Psi \left(b_{1,t+1+j} - \bar{b}_1 \right) \mathbb{E}_{t+j} \left[\beta \frac{U_c \left(c_{1,t+1+j}, l_{1,t+1+j} \right)}{U_c \left(c_{1,t+j}, l_{1,t+j} \right)} \right] \\ &= \mathbb{E}_{t+j} \left[\beta \frac{U_c \left(c_{2,t+1+j}, l_{2,t+1+j} \right)}{U_c \left(c_{2,t+j}, l_{2,t+j} \right)} \frac{rer_{t+j}}{rer_{t+1+j}} \right]. \end{aligned} \tag{36}$$

3. To break Ricardian equivalence, the representative household assumption is replaced by an overlapping generations framework based on Weil (1989) and Ghironi (2006). Each period, the population in country i grows at the constant growth rate n. As newborn households do not hold assets, the net foreign asset position is zero in the deterministic steady state. More details are provided in Appendix B.

4. The discount factor is endogenous as in Uzawa (1968), Mendoza (1991), Corsetti, Dedola, and Leduc (2008), and Schmitt-Grohé and Uribe (2003). The problem of a household is given by:

$$\max_{\substack{c_{i,t+j}, l_{i,t+j}, \\ k_{i,t+j}, x_{i,t+j}, b_{i,t+j}}} \widetilde{\mathbb{E}}_t \left\{ \sum_{j=0}^{\infty} \theta_{i,t+j} U_i \left(c_{i,t+j}, l_{i,t+j} \right) \right\} \tag{37}$$

s.t.

$$\theta_{i,t+1+j} = \beta_i \left[U \left(c_{i,t+j}, l_{i,t+j} \right) \right] \theta_{i,t+j}, \tag{38}$$

and equations (15) and (16). The discount factor is assumed to be decreasing with the level of utility, i.e., $\beta_i' (U_i) < 0$. As in Schmitt-Grohé and Uribe (2003), I lay out the case when agents do not internalize the effects of time $t+j$ choices on the discount factor in future periods. Condition (9) changes to:

$$\mathbb{E}_{t+j}\left[\beta_1\left[U\left(c_{1,t+j},l_{1,t+j}\right)\right]\frac{U_c\left(c_{1,t+1+j},l_{1,t+1+j}\right)}{U_c\left(c_{1,t+j},l_{1,t+j}\right)}\right]$$
$$=\mathbb{E}_{t+j}\left[\beta_2\left[U\left(c_{2,t+j},l_{2,t+j}\right)\right]\frac{U_c\left(c_{2,t+1+j},l_{2,t+1+j}\right)}{U_c\left(c_{2,t+j},l_{2,t+j}\right)}\frac{rer_{t+j}}{rer_{t+1+j}}\right]. \tag{39}$$

The subsequent analysis also considers the case when agents internalize the effects of their choices on the discount factor.

5. For completeness, I also investigate the case of complete international financial markets, i.e., agents have access to a full set of state-contingent claims. Efficient risk sharing implies that the marginal utilities expressed in a common good are equalized across countries subject to a constant μ_t:

$$\frac{U_c\left(c_{2,t+j},l_{2,t+j}\right)}{U_c\left(c_{1,t+j},l_{1,t+j}\right)}=rer_{t+j}\mu_t, \tag{40}$$

where μ_t is determined by the allocations and prices at time t:

$$\mu_t=\frac{rer_tU_c\left(c_{2,t},l_{2,t}\right)}{U_c\left(c_{1,t},l_{1,t}\right)}. \tag{41}$$

In the first three approaches, the wedge between expected marginal utility growth in the two countries is eliminated in a deterministic steady state only for $b_1=0$. Under endogenous discounting, condition (39) requires that $\beta_1\left[U_1\right]=\beta_2\left[U_2\right]$ in steady state. With concave utility and technology functions and β_i being strictly increasing in U_i, the allocations and prices are directly pinned down by this condition. The net foreign asset position is then determined from the goods market clearing condition in order to be compatible with these steady state allocations and prices. As, in principle, the steady state net foreign asset position may differ from zero in this model, I restrict attention to parameterizations of the endogenous discount factor such that the model implies the same steady states as the original model, i.e., net foreign assets are zero in steady state. Similarly, under complete markets the unique steady state for a given value of μ_t may not imply zero net foreign assets. Thus, I restrict attention to those value of μ_t that imply the same steady states as the original model.

4 Parameterizations

Overall, I consider five models. Model I is a simple production economy without capital and traded goods only. Model II introduces endogenous capital formation. Model III allows for capital and non-traded goods.

Table 1: Calibration of Baseline Model

Description	Symbol	Value			
1. **Parameters constant across models**					
elasticity between k and l non-traded goods (Cobb-Douglas)	κ_i^N	0.00			
elasticity between k and l traded goods (Cobb-Douglas)	κ_i^T	0.00			
depreciation rate of capital	δ_i	0.025			
elasticity between non-traded goods	ε_i^N	0.74			
elasticity between traded goods consumption	ε_i^T	0.85			
elasticity between traded goods investment	ε_i^I	0.85			
relative population size	ν_i	1.00			
discount factor	β_i	0.99			
intertemporal consumption elasticity	σ_i	2.00			
(a) *Cobb-Douglas preferences*					
weight on consumption	ξ_i	0.34			
weight on leisure	$1-\xi_i$	0.66			
(b) *Additive separable preferences*					
labor supply elasticity	χ_i	-5.00			
weight labor disutility	χ_{0i}	5.00			
2. **Parameters varying across models**					
model		I	II	III	IV	V
weight on labor in production of non-traded good	ω_{li}^N	n.a.	n.a.	n.a.	0.560	0.560
weight on labor in production of traded good	ω_{li}^T	1.000	0.610	0.610	0.610	0.610
distribution cost in terms of non-traded good	η_i	n.a.	n.a.	0.000	1.090	1.090
(a) *Cobb-Douglas preferences*						
weight own traded good in consumption	α_{ii}^T	0.938	0.938	0.910	0.720	0.910
weight own traded good in investment	α_{ii}^I	1.000	0.938	0.910	1.000	0.855
weight overall traded good	α_{i1}^N	1.000	1.000	0.550	0.550	0.550
(b) *Additive separable preferences*						
weight own traded good in consumption	α_{ii}^T	0.945	0.945	0.924	0.720	0.905
weight own traded good in investment	α_{ii}^I	1.000	0.945	0.924	1.000	0.905
weight overall traded good	α_{i1}^N	1.000	1.000	0.550	0.550	0.550

The fourth model is identical to the model presented in Corsetti, Dedola, and Leduc (2008), i.e., consumption of traded goods requires the use of non-traded distribution services. In contrast to models II and III, investment features complete home bias. Relative to model IV, model V relaxes the assumption of complete home bias in investment. As models IV and V feature non-zero distribution costs, the trade elasticity is not fully determined by the elasticity of substitution between traded goods.

Table 1 summarizes the parameterizations for the five different models. To the extent possible, parameters are set equal across models. Parameter differences reflect the differences in model features. For each model, the quasi shares in equation (21) and (22) are adjusted to obtain an import to GDP ratio of 6.2 percent as in Corsetti, Dedola, and Leduc (2008).

Household preferences are described either by a Cobb-Douglas function:

$$U_i(c_{i,t+j}, l_{i,t+j}) = \frac{\left(c_{i,t+j}^{\xi_i}\left(1 - l_{i,t+j}\right)^{1-\xi_i}\right)^{1-\sigma_i}}{1 - \sigma_i}, \tag{42}$$

or an additive separable utility function:

$$U_i(c_{i,t+j}, l_{i,t+j}) = \frac{c_{i,t+j}^{1-\sigma_i}}{1 - \sigma_i} - \chi_{0i}\frac{l_{i,t+j}^{1-\chi_i}}{1 - \chi_i}. \tag{43}$$

The functional forms for the stationarity inducing devices are:

- for the portfolio costs approach

$$\Gamma(b_{i,t+j}) = \frac{1}{2}\gamma_i\left(\frac{b_{i,t+j}}{P_{i,t+j}}\right)^2 P_{i,t+j}, \tag{44}$$

- for a debt-elastic interest rate

$$\Psi(b_{i,t+j}) = \exp\left(\psi_d\left(b_{i,t+j} - \bar{b}\right)\right), \tag{45}$$

- for endogenous discounting under Cobb-Douglas preferences

$$\beta_{i,t+j} = \left(1 + \psi_i c_{i,t+j}^{\xi_i}(1 - l_{i,t+j})^{1-\xi_i}\right)^{-1}, \tag{46}$$

  under additive separable preferences

$$\beta_{i,t+j} = 1 - \psi_i \exp\left(\frac{c_{i,t+j}^{1-\sigma_i}}{1 - \sigma_i} - \chi_{0i}\frac{l_{i,t+j}^{1-\chi_i}}{1 - \chi_i}\right). \tag{47}$$

5 Steady state multiplicity

I first compute the set of steady states under the assumption that financial markets are absent at the international level, i.e., $b_{1,t+j} = 0$ for all j, before commenting on the remaining market arrangements outlined in Section 3.2.

Absent dynamics all the endogenous variables can be expressed as functions of the relative price $q = \frac{P_2}{P_1}$. An equilibrium with zero net foreign asset holdings is then characterized by any value of q that implies zero excess demand z_2 for the traded good of country 2:

$$z_2(q) \equiv (c_{12}(q) + x_{12}(q))\,\nu_1 + \left(c_{22}(q) + x_{22}(q) - y_2^T(q)\right)\nu_2 \tag{48}$$

$$z_2(q) \leq 0,\ \infty \geq q \geq 0 \text{ and } qz_2(q) = 0. \tag{49}$$

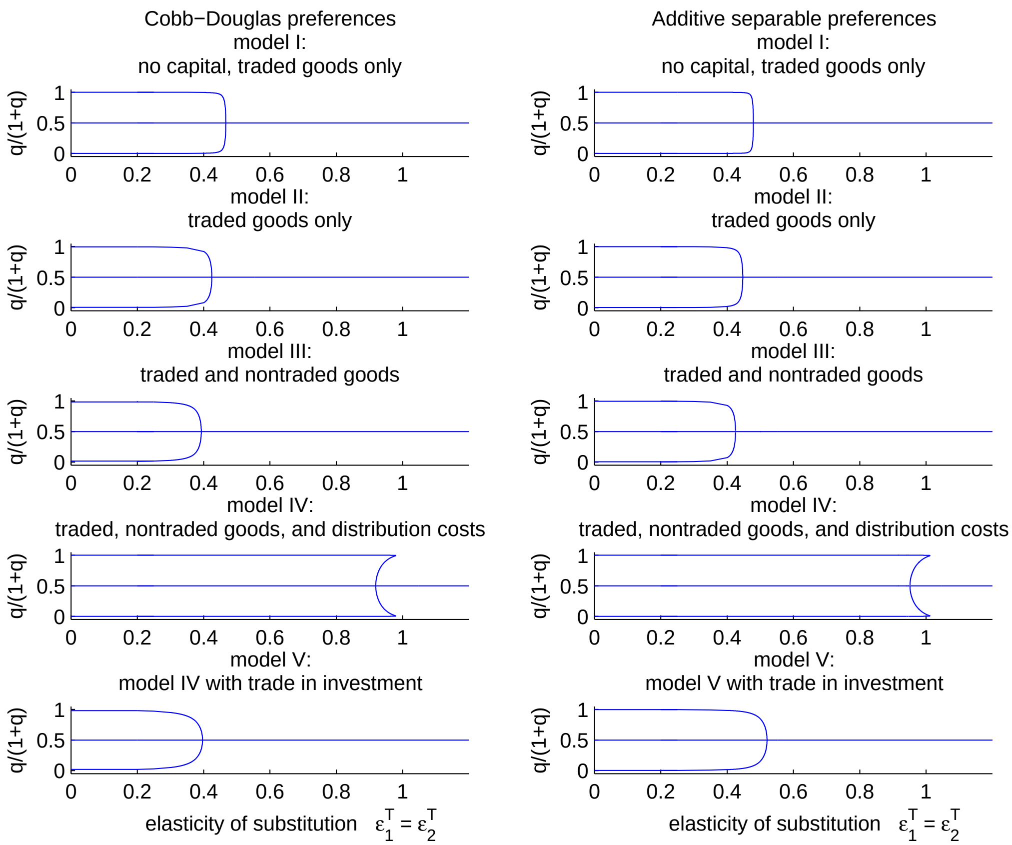
Thus, the steady states can be computed reliably by searching over a one-dimensional grid on the interval $[0, 1]$ for the normalized relative price $\frac{q}{1+q}$.

5.1 Steady states under financial autarchy

Figure 2 plots the set of steady state equilibrium prices for different values of the elasticity of substitution between traded goods in the five models for both Cobb-Douglas preferences and additive separable preferences. Each model displays a unique price equilibrium with $q = 1$ $\left(\frac{q}{1+q} = \frac{1}{2}\right)$, if the substitution elasticity is high. When the two traded goods are assumed to be less substitutable, multiple steady states arise. The two asymmetric steady states feature a relative price $\frac{q}{1+q}$ close to zero or one, respectively, and are hard to detect in practice.

To understand how multiple equilibria arise at low values of the elasticity of substitution consider the case of the model without capital and traded goods only (model I). If the price of good 1 is high relative to the price of good 2 (lower branch), the value of country 1's production is high relative to country 2's. As labor supply is assumed to be fairly inelastic, overall production is price inelastic, as well. Agents in country 2 are willing to pay the high price for good 1 and country 1 ends up consuming most of the two goods. Similar logic applies when q is much larger than 1 (upper branch) with the roles of countries 1 and 2 being reversed. The middle equilibrium, which always exists, is the symmetric equilibrium with $q = 1$.

Figure 2: Steady state multiplicity for different values of the elasticity of substitution



Notes: Set of normalized relative prices $q/(1+q)$ that constitute a steady state absent financial markets as a function of the elasticity of substitution between traded goods. The (symmetric) parameterizations of models I to V are given in Table 1.

As can be seen in Figure 2 and Table 2 the threshold value of the substitution elasticity $\bar{\varepsilon}_1^T$ for which equilibrium multiplicity occurs is model dependent, but it does not vary much across the two specifications of household preferences for standard parameterizations. Keeping trade shares constant at an import to GDP ratio of 6.2 percent across models, $\bar{\varepsilon}_1^T$ falls when capital accumulation and non-traded goods are introduced. For model II, with an endogenous capital stock, a country carries a lower capital stock and reduces production, if the relative price for its good is low. Given reduced supply of the country's good, the low relative price may turn out not to be an equilibrium. Hence, for equilibrium multiplicity to occur, $\bar{\varepsilon}_1^T$ has to be lower in the model with an endogenous capital stock than in model I. In model III, the presence of non-traded goods opens up an additional channel for a country to react to a low relative price of its traded good. By shifting labor and capital towards the production of the non-traded good, production of the traded good is lowered, making it even less likely that the low relative price constitutes an equilibrium for given ε_1^T.

The two bottom panels of Figure 2 introduce distribution costs as in Corsetti, Dedola, and Leduc (2008). For model IV, which is parameterized exactly as in Corsetti, Dedola, and Leduc (2008), the threshold value $\bar{\varepsilon}_1^T$ is well above its value in models I-III. However, as consumption of traded goods requires non-traded goods as input, the implied trade elasticity is 0.5359 for Cobb-Douglas preferences and 0.5532 for additive separable preferences.[9] Furthermore, there are five steady states under Cobb-Douglas preferences for $\varepsilon_1^T \in [0.9189; 0.9798]$ and three steady states for $\varepsilon_1^T \in [0; 0.9189]$. Under additive separable preferences, there are five steady states for $\varepsilon_1^T \in [0.9516; 1.0115]$ and three steady states for $\varepsilon_1^T \in [0; 0.9516]$.

Table 2: Model dependent threshold levels for $\bar{\varepsilon}_1^T = \bar{\varepsilon}_2^T$

Model	I	II	III	IV	V
Cobb-Douglas pref.	0.4670	0.4248	0.3930	0.9798	0.3968
Add. sep. pref.	0.4793	0.4474	0.4255	1.0115	0.5205

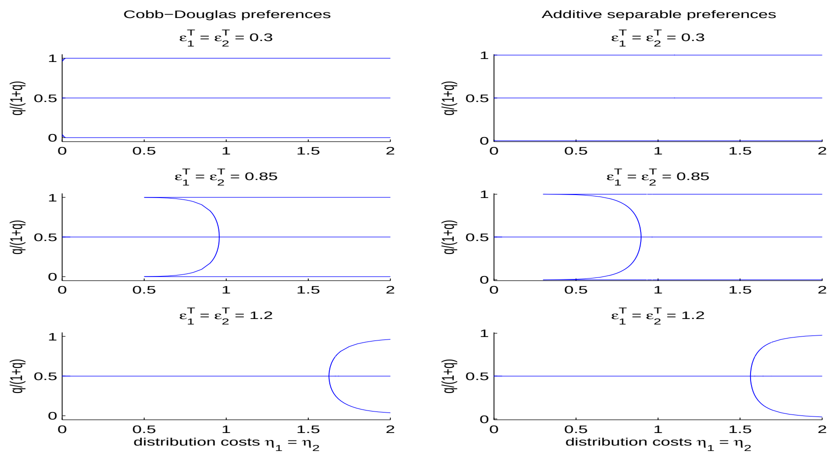
Whereas the baseline model in Corsetti, Dedola, and Leduc (2008) assumes complete home bias in investment, the bottom panel of the figure assumes incomplete home bias in investment. Unlike for consumption, no distribution services are required for the investment good. In model V, the threshold value $\bar{\varepsilon}_1^T$ falls well below unity again. Furthermore, the maximum number of equilibria is three. If investment was subject to sufficiently high distribution services, however, a situation similar to model IV would reemerge.

Figure 3 further explores the role of distribution costs in generating multiple steady states. Starting from the parameterization of model IV, the figure shows how the steady state values of the relative price q change as distribution costs are raised for a fixed value of ε_1^T. For low

[9] Numerically, the steady state value of the prices of the non-traded goods do not vary with ε_1^T. While in equation (32) the measured trade elasticity in the symmetric steady state is proportional to the elasticity of substitution between traded goods, this is not true for the asymmetric steady states.

values of ε_1^T, such as in the top panel, multiple steady states exist even absent distribution costs. Irrespective of the value of ε_1^T assumed in the figure, multiple steady states arise once distribution costs are sufficiently high.

Figure 3: Steady state multiplicity in model IV (CDL) for changing distribution costs



Notes: Set of normalized relative prices $q/(1+q)$ that constitute a steady state absent financial markets as a function of the distribution cost parameters. Except for the elasticity of substitution between traded goods and the distribution cost parameters, parameters are chosen to match the calibration in Corsetti, Dedola, and Leduc (2008).

5.2 Steady states under different assumptions on international financial markets

The choice of the stationarity inducing device affects the number of steady state equilibria. In a steady state with zero net foreign assets, the standard incomplete markets model, the portfolio cost model, and the debt-elastic interest rate model imply the same steady state restrictions as the model without international financial markets. If under financial autarchy multiple values of the relative price q induce market clearing, then this is also the case for these three frameworks. Similarly, the overlapping generations model with incomplete markets displays the same steady states as its analogue without financial markets.[10]

[10]As shown in Appendix B, the steady states in the overlapping generations model differ from the steady states in models for which each country's agents can be represented by a single agent. However, for the parameterizations adopted in this paper, the quantitative differences are small.

By contrast, for a given parameterization of the discount factor functions, the model with endogenous discounting admits a unique steady state irrespective of the number of steady states under financial autarchy. Condition (39) determines uniquely the steady state allocations and prices including the relative price q. It is not guaranteed that the resulting steady state coincides with any of the steady states obtained under financial autarchy. I restrict the set discount factor functions such that the unique steady state under endogenous discounting coincides with one of the steady states obtained under financial autarchy and thus features a zero net foreign asset position.

Similarly, under complete markets, for a given initial value of μ_t, condition (40) admits only one steady state irrespective of the number of steady states in the model with financial autarchy. As each value of μ_t gives rise to a different steady state, that may not coincide with any of those obtained under financial autarchy, I restrict attention to those values of μ_t, that imply a steady state that can also be obtained under financial autarchy.

6 Local equilibrium dynamics

Turning to the dynamics of the different models in the neighborhood of a given deterministic steady state, the model is (log-)linearized around this steady state. Reducing the system of equilibrium conditions to its state variables (log deviations of the capital stocks $\hat{k}_{1,t}$ and $\hat{k}_{2,t}$, and the absolute deviation of the net foreign asset position $\Delta b_{1,t}$), one obtains the difference equation:

$$a \begin{pmatrix} z_{t+1} \\ z_t \end{pmatrix} + b \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix} = 0,$$

with the 6×6 coefficient matrices a and b and the 3×1 state vector $z_t = \left(\Delta b_{1,t}, \hat{k}_{1,t}, \hat{k}_{2,t}\right)'$. With a being an invertible matrix for the models considered in this paper, a simple count of the eigenvalues of the matrix $-a^{-1}b$ suffices to characterize the stability of the local dynamics as described in Christiano (2002).[11] Let N (here equal 6) denote the number of eigenvalues of

[11] Christiano (2002) discusses also the case if a is not invertible.

$-a^{-1}b$ and $N^* \leq N$ be the number of eigenvalues larger than one in absolute value. n (here equal to 3) is the number of predetermined variables. The equilibrium dynamics identified by the minimum state variable solution are:

1. bounded and unique, if $N^* = N - n$,
2. bounded, but not unique, if $N^* < N - n$,
3. unbounded, if $N^* > N - n$.

Thus, the equilibrium dynamics can be characterized by the absolute value of the nth and $n+1$st smallest eigenvalue, $|\lambda_n|$ and $|\lambda_{n+1}|$. If $|\lambda_n| > 1$, the equilibrium dynamics are unbounded. If $|\lambda_n| < 1$ and $|\lambda_{n+1}| > 1$, the dynamics are bounded and unique. Otherwise, they are bounded, but not unique. As $|\lambda_{n+1}| > 1$ for all cases analyzed below, I will reduce attention to $|\lambda_n|$. The subsequent analysis also links $|\lambda_n|$ to the slope of the excess demand function for good 2 with respect to the relative price $\frac{\partial z_2}{\partial q}q$ in a steady state.

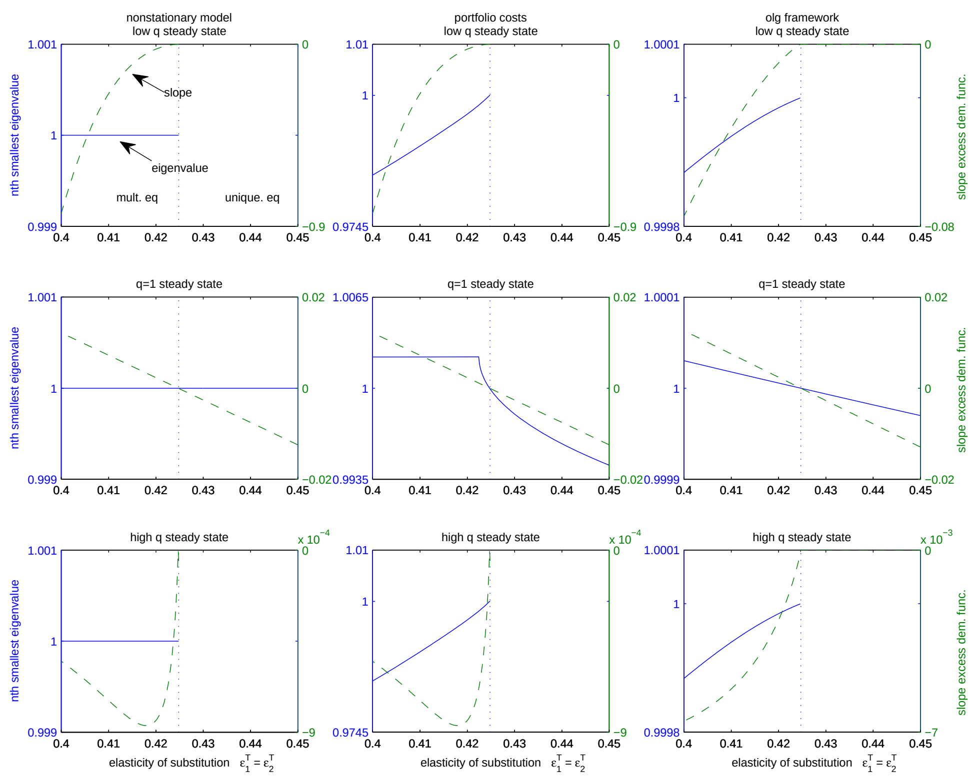
To study the relationship between the stability of the equilibrium dynamics around a steady state and the multiplicity of steady states, I analyze the model with endogenous capital and traded goods (model II) as well as the model with distribution costs (model IV) under different stationarity inducing devices. In the following, ε_1^T and ε_2^T are always set to be equal.

6.1 Results for model II

Figures 4 and 5 perform this stability analysis for model II. Figure 4 examines the cases of standard incomplete markets, portfolio costs, and overlapping generations. For each value of the elasticity of substitution $\varepsilon_1^T \in [0.4, 0.45]$, the model is linearized around each of the steady states that exist for ε_1^T. The value of the nth smallest eigenvalue of the system $|\lambda_n|$ (left scale) and the slope of the excess demand function for good 2 in the steady state (right scale) are recorded. In accordance with the second panel in Figure 2, there are three steady states for $\varepsilon_1^T < \bar{\varepsilon}_1^T = 0.4248$ in the standard incomplete markets model and the portfolio cost model, but a unique one otherwise. For the overlapping generations model the value is $\bar{\varepsilon}_1^T = 0.4247$. The label "low q steady state" refers to steady states represented by the lower branch in Figure 2 for model II with Cobb-Douglas preferences. "High q steady state" and "$q = 1$ steady state" refer to the top and middle branches, respectively.

Under standard incomplete markets (left column) $|\lambda_n|$ equals unity irrespective of both the steady state around which the model is approximated and the value of ε_1^T. This finding simply restates the non-stationarity of the net foreign asset position.

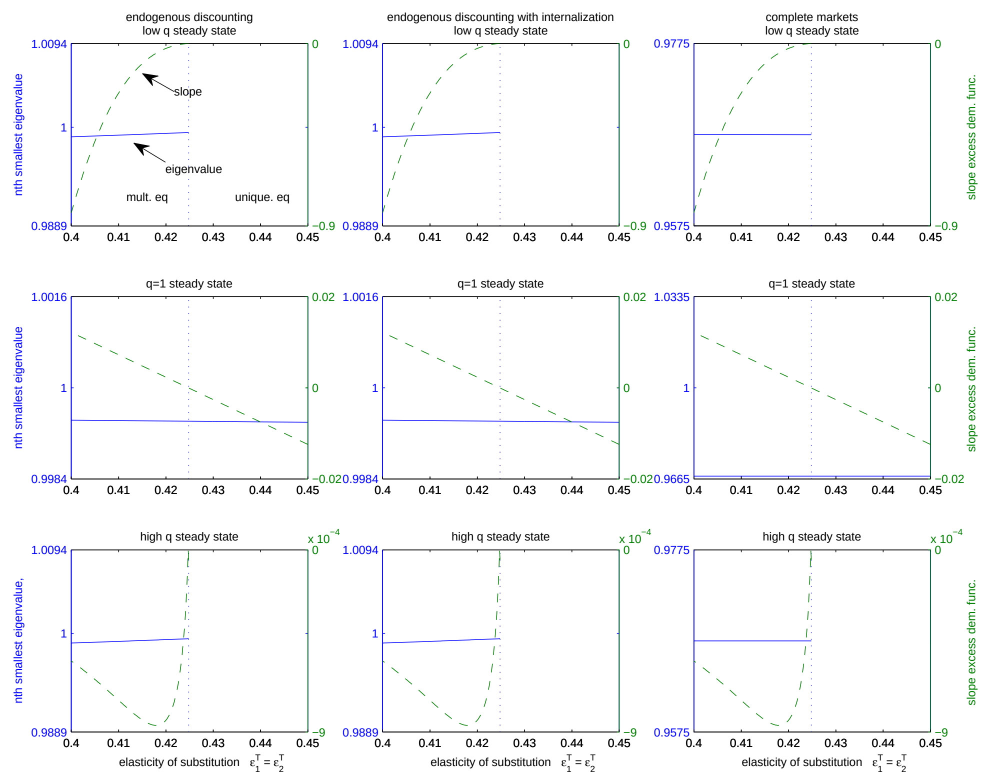
Figure 4: Local dynamics in model II (BKK) unstable models



Notes: "Low q steady state" refers to steady states represented by the lower branch in Figure 2 for model II with Cobb-Douglas preferences. "High q steady state" and "$q = 1$ steady state" refer to the top and middle branch, respectively. Each panel shows the nth largest eigenvalue (left axis) and the slope of the excess demand function (right axis) associated with the steady state around which the model is approximated as a function of the elasticity of substitution between traded goods. For all three models n equals 3 and $N = 6$. If a steady state does not exist for a given range of the substitution elasticity, no values are plotted.

For the portfolio cost model (middle column), in the top and bottom panels, $|\lambda_n|$ is always smaller than 1 and the equilibrium dynamics are therefore locally unique and bounded. In line with Section 2, the slope of the excess demand function is always negative for these steady states. However, in the middle panel, that shows the results for the symmetric steady state of the model, it is $|\lambda_n| < 1$ with $\frac{\partial z_2}{\partial q} q > 0$ for $\varepsilon_1^T > \bar{\varepsilon}_1^T$, and it is $|\lambda_n| \geq 1$ with $\frac{\partial z_2}{\partial q} q \leq 0$ for $\varepsilon_1^T \leq \bar{\varepsilon}_1^T$. Hence, for $\varepsilon_1^T \leq \bar{\varepsilon}_1^T$ the equilibrium dynamics around the symmetric steady state are not bounded. Although not shown, the same findings apply for the case of a debt-elastic interest rate.

Figure 5: Local dynamics in model II (BKK) stable models



Notes: "Low q steady state" refers to steady states represented by the lower branch in Figure 2 for model II with Cobb-Douglas preferences. "High q steady state" and "$q = 1$ steady state" refer to the top and middle branch, respectively. Each panel shows the nth largest eigenvalue (left axis) and the slope of the excess demand function (right axis) associated with the steady state around which the model is approximated as a function of the elasticity of substitution between traded goods. For the first two models n equals 3 and $N = 6$, for the complete markets model n=2 and $N = 4$. If a steady state does not exist for a given range of the substitution elasticity, no values are plotted.

In the overlapping generations framework (right column) the same patterns arise: steady states for which the excess demand function is downward sloping are associated with $|\lambda_n| < 1$ and locally unique and bounded dynamics. Otherwise, the equilibrium dynamics are unbounded.

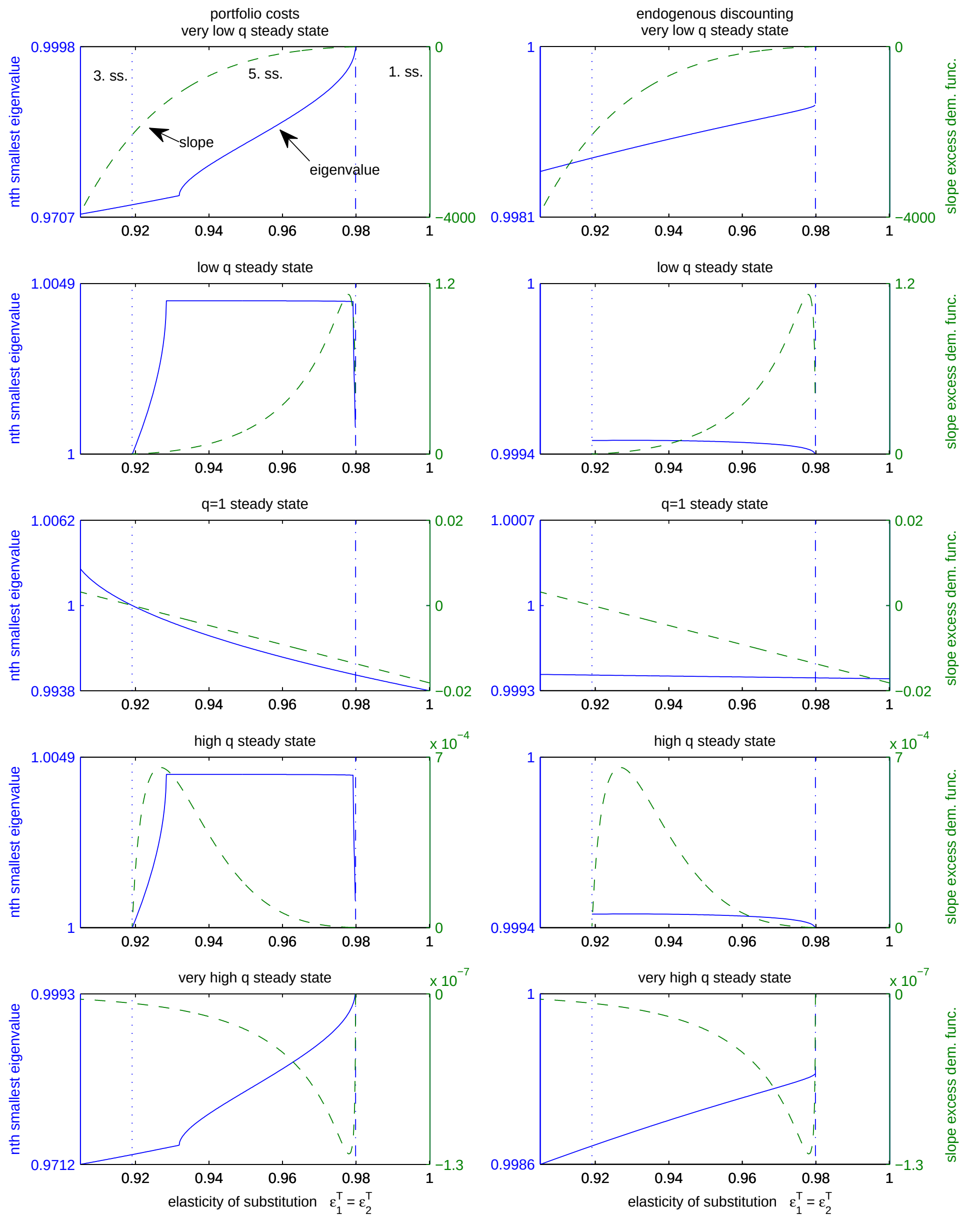
The local dynamics under endogenous discounting – both with and without internalization – and under complete markets are studied in Figure 5. As pointed out earlier, for a given parameterization of the discount factor function or the choice of μ_t, these three settings always imply a unique steady state. The figure, though, shows results for each of the steady states that occur under financial autarchy by appropriately adjusting the discount factor function or the weight μ_t, respectively. The first two columns reveal that $|\lambda_n|$ is always less than unity irrespective of the steady state that is analyzed and the associated sign of $\frac{\partial z_2}{\partial q} q$. Thus, under endogenous discounting, the equilibrium dynamics are always unique and bounded. The same findings apply under complete markets.

6.2 Results for model IV

The dynamic characteristics of model IV under the calibration of Corsetti, Dedola, and Leduc (2008) with Cobb-Douglas preferences are examined in Figure 6 for the case of portfolio costs and endogenous discounting. Each of the five panels per column corresponds to one of the steady states depicted in the fourth panel of Figure 2.

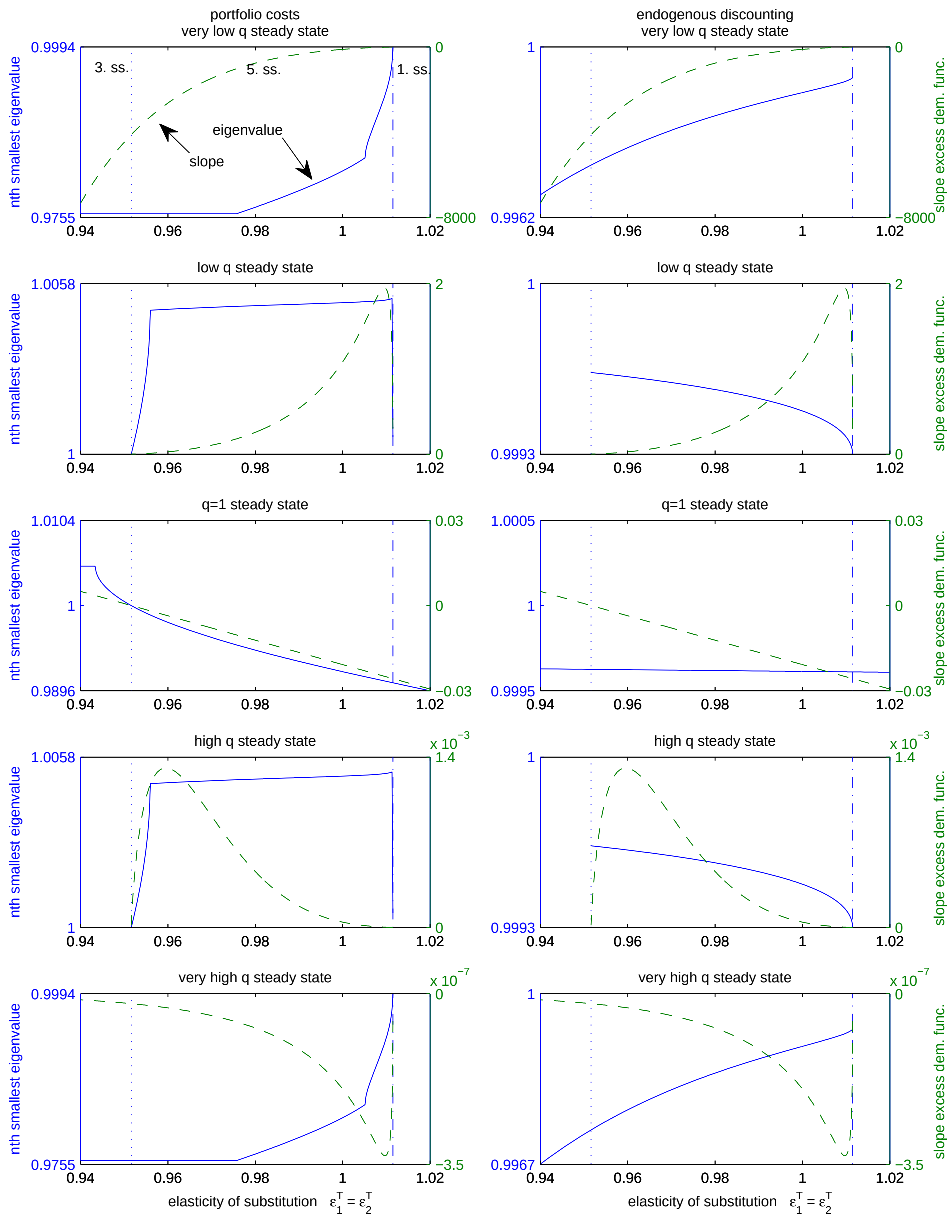
For the portfolio cost model, $|\lambda_n|$ is always less than unity in the first and fifth panels and $\frac{\partial z_2}{\partial q} q < 0$. To the extent that these steady states exist, which is the case for $\varepsilon_1^T < 0.979$, these steady states are associated with unique and bounded equilibrium dynamics. For the second and fourth panels, $|\lambda_n|$ is always larger than unity and the equilibrium dynamics are unbounded. Consistent with all previous findings, the sign of $\frac{\partial z_2}{\partial q} q$ is positive. The case of $q = 1$ (third panel) is more interesting. For $\varepsilon_1^T > 0.9189$, $\frac{\partial z_2}{\partial q} q$ is negative and $|\lambda_n| < 1$. It is only for $\varepsilon_1^T < 0.9189$ that the excess demand function switches sign, $|\lambda_n|$ turns larger than one, and the equilibrium dynamics become unbounded. Under endogenous discounting $|\lambda_n|$ is always smaller than one irrespective of the steady state around which the model is approximated and the sign of $\frac{\partial z_2}{\partial q} q$. The equilibrium dynamics are unique and bounded.

Figure 6: Local dynamics in model IV (CDL) with Cobb-Douglas preferences



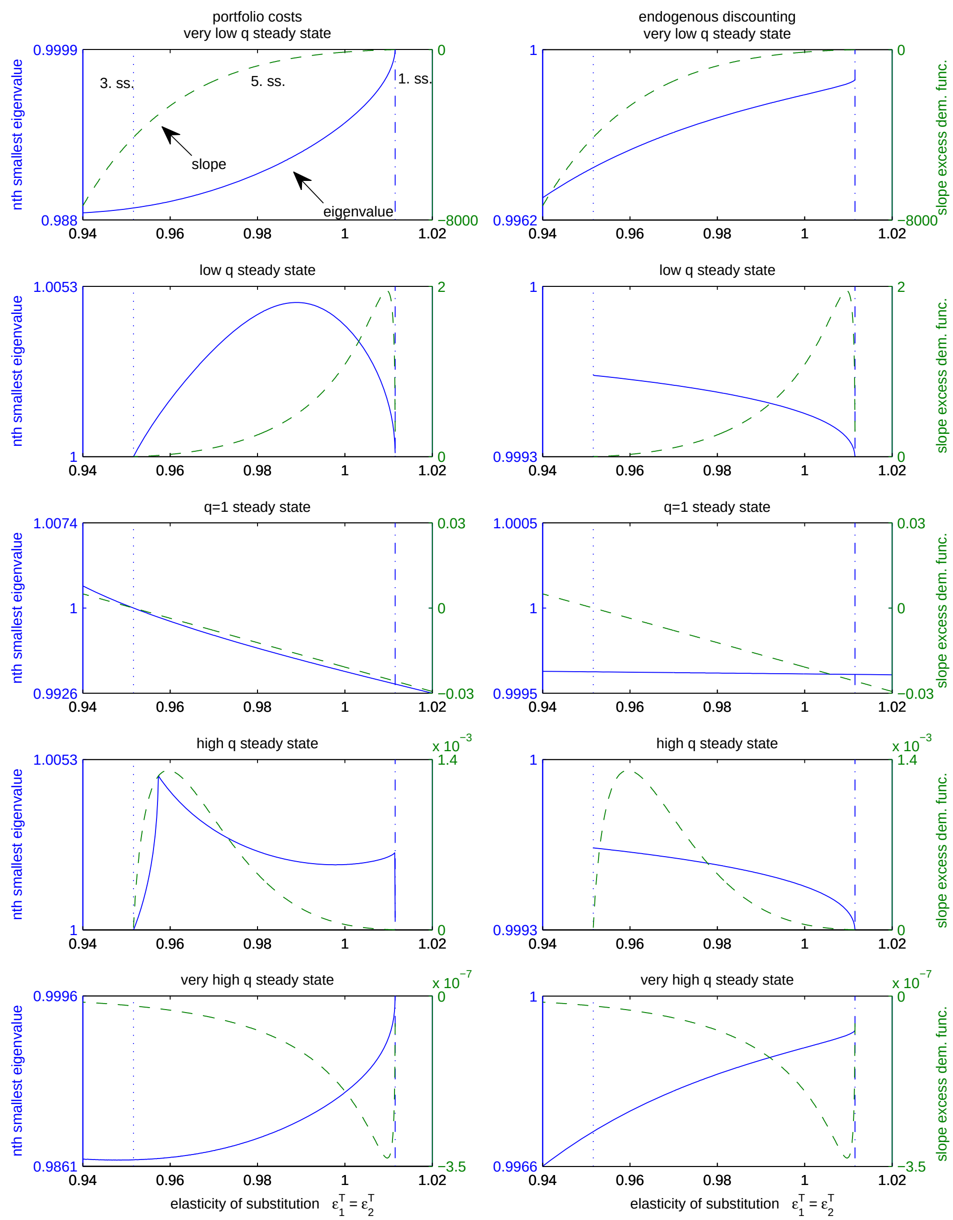
Notes: The labels "very low q steady state", "low q steady state", "$q = 1$ steady state", "high q steady state", and "very high q steady state" refer to the steady states represented by the bottom, second from bottom, middle, second from top, and top branch in Figure 2 for model IV with Cobb-Douglas preferences, respectively. Each panel shows the nth largest eigenvalue (left axis) and the slope of the excess demand function (right axis) associated with the steady state around which the model is approximated as a function of the elasticity of substitution between traded goods. For both models n equals 3 and $N = 6$. If a steady state does not exist for a given range of the substitution elasticity, no values are plotted. The steady state is unique for $\varepsilon_1^T > 0.9798$. There are five steady states for $\varepsilon_1^T \in [0.9189; 0.9798]$ and three steady states for $\varepsilon_1^T < 0.9189$.

Figure 7: Local dynamics in model IV (CDL) with additive separable preferences



Notes: The labels "very low q steady state", "low q steady state", "$q = 1$ steady state", "high q steady state", and "very high q steady state" refer to the steady states represented by the bottom, second from bottom, middle, second from top, and top branch in Figure 2 for model IV with additive separable preferences, respectively. Each panel shows the nth largest eigenvalue (left axis) and the slope of the excess demand function (right axis) associated with the steady state around which the model is approximated as a function of the elasticity of substitution between traded goods. For both models n equals 3 and $N = 6$. If a steady state does not exist for a given range of the substitution elasticity, no values are plotted. The steady state is unique for $\varepsilon_1^T > 1.0115$. There are five steady states for $\varepsilon_1^T \in [0.9516; 1.0115]$ and three steady states for $\varepsilon_1^T < 0.9516$.

Figure 8: Local dynamics in model IV (CDL) with additive separable preferences, real and nominal rigidities



Notes: The labels "very low q steady state", "low q steady state", "$q = 1$ steady state", "high q steady state", and "very high q steady state" refer to the steady states represented by the bottom, second from bottom, middle, second from top, and top branch in Figure 2 for model IV with additive separable preferences, respectively. Panels show the nth largest eigenvalue (left axis) and the slope of the excess demand function (right axis) associated with the steady state around which the model is approximated as a function of the elasticity of substitution between traded goods. Due to the additional state variables n equals 40 and $N = 78$. The steady state is unique for $\varepsilon_1^T > 1.0115$. There are five steady states for $\varepsilon_1^T \in [0.9516; 1.0115]$ and three steady states for $\varepsilon_1^T < 0.9516$. Monetary policy follows a Taylor-type rule with a long-run weight on consumer price inflation of 1.5, an interest rate smoothing coefficient of 0.7, and zero weight on the output gap.

Allowing for additive separable preferences does not overturn these findings as shown in Figure 7. Introducing real and nominal rigidities under additive separable preferences does not affect the findings either. Figure 8 confirms this claim in a model with Calvo sticky wage and price contracts, consumption habits, investment adjustment costs, and local currency pricing. Monetary policy follows an interest rate rule with the policy rate being a function of consumer price inflation only. A detailed model description is provided in the Technical Appendix accompanying this paper.

7 Conclusions

Widely used international business cycle models admit multiple steady states absent international financial markets for low substitutability between traded goods. Once a limited set of internationally traded assets is introduced, the steady state multiplicity may or may not be preserved. If the incomplete markets model is augmented with portfolio costs, a debt-elastic interest rate, or an overlapping generations framework as in Weil (1989) the number of steady states remains unchanged relative to the model without financial markets. If Uzawa-type preferences are introduced (Uzawa (1968)), the steady state of the model is unique.

The choice of stationarity inducing device also affects the stability of the dynamics around a given steady state. Under portfolio costs, debt-elastic interest rates, or overlapping generations, steady states that are associated with a downward-sloping excess demand function of the foreign good display locally unique and bounded dynamics. For steady states with an upward-sloping excess demand function, the local dynamics are unbounded. Under endogenous discounting or complete markets, the local dynamics are always unique and bounded.

The results stressed in this paper prevail under (local) higher order perturbation methods. The findings also extend to environments with a larger set of available assets. If country portfolios are determined as in Devereux and Sutherland (2008) and Tille and van Wincoop (2010), the net foreign asset position is not determined in steady state and displays unit root behavior unless stationarity is induced with one of the devices discussed in this paper.

Absent trade adjustment costs short- and long-run substitution elasticities are identical in this paper. If trade adjustment costs were to affect the model's allocations away from a steady state, the analysis would not change given the use of local solution techniques. Steady state multiplicity would only occur if the long-run substitution elasticity between traded goods fell below its threshold level $\bar{\varepsilon}_i^T$ in the model without trade adjustment costs. However, if one were to employ global solution techniques, differences across models could be detected. Similar to the findings presented in Bodenstein (2010) for the case of capital, a lower short-run substitution elasticity may lead to multiple equilibrium paths despite a unique steady state for the given value of the long-run substitution elasticity.

References

Backus, D., P. Kehoe, and F. Kydland (1995). International business cycles: theory and evidence. In T. Cooley (Ed.), *Frontiers of Business Cycle Research*, pp. 331–356. Princeton University Press.

Backus, D. K. and G. W. Smith (1993). Consumption and real exchange rates in dynamic exchange economies with non-traded goods. *Journal of International Economics 35*(3-4), 297–316.

Baxter, M. and M. J. Crucini (1995). Business cycles and the asset structure of foreign trade. *International Economic Review 36*(4), 821–54.

Benigno, G. and C. Thoenissen (2008). Consumption and real exchange rates with incomplete markets and non-traded goods. *Journal of International Money and Finance 27*(6), 926–948.

Bodenstein, M. (2010). Trade elasticity of substitution and equilibrium dynamics. *Journal of Economic Theory 145*(3), 1033–1059.

Boileau, M. and M. Normandin (2008). Closing international real business cycle models with restricted financial markets. *Journal of International Money and Finance 27*(5), 733–756.

Broda, C. and D. E. Weinstein (2006). Globalization and the gains from variety. *The Quarterly Journal of Economics 121*(2), 541–585.

Burstein, A. T., J. C. Neves, and S. Rebelo (2003). Distribution costs and real exchange rate dynamics during exchange-rate-based stabilizations. *Journal of Monetary Economics 50*(6), 1189–1214.

Chari, V. V., P. J. Kehoe, and E. R. McGrattan (2002). Can sticky price models generate volatile and persistent real exchange rates? *Review of Economic Studies 69*(3), 533–63.

Christiano, L. J. (2002). Solving dynamic equilibrium models by a method of undetermined coefficients. *Computational Economics 20*(1-2), 21–55.

Collard, F. and H. Dellas (2007). Technology shocks and employment. *Economic Journal 117*(523), 1436–1459.

Corsetti, G. and L. Dedola (2005). A macroeconomic model of international price discrimination. *Journal of International Economics 67*(1), 129–155.

Corsetti, G., L. Dedola, and S. Leduc (2008). International risk sharing and the transmission of productivity shocks. *Review of Economic Studies 75*(2), 443–473.

Devereux, M. B. and G. W. Smith (2007). Transfer problem dynamics: macroeconomics of the franco-prussian war indemnity. *Journal of Monetary Economics 54*(8), 2375–2398.

Devereux, M. B. and A. Sutherland (2008). Financial globalization and monetary policy. *Journal of Monetary Economics 55*(8), 1363–1375.

Ghironi, F. (2006). Macroeconomic interdependence under incomplete markets. *Journal of International Economics 70*(2), 428–450.

Ghironi, F. (2008). The role of net foreign assets in a new keynesian small open economy model. *Journal of Economic Dynamics and Control 32*(6), 1780–1811.

Heathcote, J. and F. Perri (2002). Financial autarky and international business cycles. *Journal of Monetary Economics 49*(3), 601–627.

Hooper, P., K. Johnson, and J. Marquez (1998). Trade elasticities for g-7 countries. International Finance Discussion Papers 609, Board of Governors of the Federal Reserve System (U.S.).

Kehoe, T. J. (1991). Computation and multiplicity of equilibria. In W. Hildenbrand and H. Sonnenschein (Eds.), *Handbook of Mathematical Economics*, Volume 4 of *Handbook of Mathematical Economics*, Chapter 38, pp. 2049–2144. Elsevier.

Kim, S. and M. Kose (2003). Dynamics of open-economy business-cycle models: role of the discount factor. *Macroeconomic Dynamics 7*(2), 263–290.

Kollmann, R. (2006, February). International portfolio equilibrium and the current account. CEPR Discussion Papers 5512, C.E.P.R. Discussion Papers.

Lubik, T. and F. Schorfheide (2006). A bayesian look at the new open economy macroeconomics. In *NBER Macroeconomics Annual 2005, Volume 20*, NBER Chapters, pp. 313–382. National Bureau of Economic Research, Inc.

Mas-Colell, A., M. Whinston, and J. Green (1995). *Microeconomic Theory.* Oxford University Press.

Mendoza, E. G. (1991). Real business cycles in a small open economy. *American Economic Review 81*(4), 797–818.

Rabanal, P. and V. Tuesta (2010). Euro-dollar real exchange rate dynamics in an estimated two-country model: An assessment. *Journal of Economic Dynamics and Control 34*(4), 780–797.

Schmitt-Grohé, S. and M. Uribe (2003). Closing small open economy models. *Journal of International Economics 61*(1), 163–185.

Stockman, A. C. and L. L. Tesar (1995). Tastes and technology in a two-country model of the business cycle: explaining international comovements. *American Economic Review 85*(1), 168–85.

Taylor, J. (1993). *Macroeconomic policy in a world economy: from economic design to practical operation.* Norton, New York, NY.

Tille, C. and E. van Wincoop (2010). International capital flows. *Journal of International Economics 80*(2), 157–175.

Uzawa, H. (1968). Value, capital and growth: papers in honor of sir john hicks. In J. Wolfe (Ed.), *Time preference, the consumption function and optimum asset holdings*, pp. 485–504. The University of Edinburgh Press.

Weil, P. (1989). Overlapping families of infinitely-lived agents. *Journal of Public Economics 38*(2), 183–198.

Whalley, J. (1984, June). *Trade Liberalization among Major World Trading Areas*, Volume 1 of *MIT Press Books.* The MIT Press.

A Appendix: Details on the simple model

This appendix describes the equations underlying the discussion in Section 2. The first order conditions of the households imply:

$$c_{12,t} = \frac{\alpha_{12}^T q_t^{-\varepsilon^T}}{\alpha_{11}^T + \alpha_{12}^T q_t^{1-\varepsilon^T}} \left[y_{1,t}^T + \frac{b_{1,t-1} - Q_{1,t} b_{1,t}}{\Phi_1(q_t)} \right], \tag{50}$$

$$c_{22,t} = \frac{\alpha_{22}^T}{\alpha_{21}^T \left(\frac{1}{q_t}\right)^{1-\varepsilon^T} + \alpha_{22}^T} \left[y_{2,t}^T + \frac{b_{2,t-1} - Q_{2,t} b_{2,t}}{\Phi_1(q_t) q_t} \right], \tag{51}$$

$$c_{1,t}^T = \Phi_1(q_t) y_{1,t}^T + b_{1,t-1} - Q_{1,t} b_{1,t}, \tag{52}$$

$$c_{2,t}^T = \Phi_2(q_t) y_{2,t}^T + \frac{\Phi_2(q_t)}{q_t \Phi_1(q_t)} \left(b_{2,t-1} - Q_{2,t} b_{2,t} \right), \tag{53}$$

where

$$\Phi_1(q_t) \equiv \frac{P_{1,t}}{P_{1,t} c_{11,t} + P_{2,t} c_{12,t}} = \left[\alpha_{11}^T \left(\frac{1}{q_t}\right)^{1-\varepsilon^T} + \alpha_{12}^T \right]^{-\frac{1}{1-\varepsilon^T}} \frac{1}{q_t}, \tag{54}$$

$$\Phi_2(q_t) \equiv \frac{P_{2,t}}{P_{1,t} c_{21,t} + P_{2,t} c_{22,t}} = \left[\alpha_{21}^T \left(\frac{1}{q_t}\right)^{1-\varepsilon^T} + \alpha_{22}^T \right]^{-\frac{1}{1-\varepsilon^T}}, \tag{55}$$

$$rer_t = q_t \frac{\Phi_1(q_t)}{\Phi_2(q_t)}. \tag{56}$$

The price of the aggregate consumption good in country 1 is taken to be the numéraire. Log-linearizing these equations around a steady state with zero net foreign assets and setting the exogenous variables $y_{1,t}^T$ and $y_{2,t}^T$ to be constant ("$\hat{x}$" for log-linearized and "Δx" for linearized variables x). Variables without time index indicate steady state values:

$$\hat{c}_{12,t} = -\left\{ (1-\varepsilon^T)(1-\alpha_{11}^T) + \varepsilon^T \right\} \hat{q}_t + \frac{1}{y_1 \Phi_1(q)} \left[\Delta b_{1,t-1} - \beta \Delta b_{1,t} \right], \tag{57}$$

$$\hat{c}_{22,t} = (1-\varepsilon^T)(1-\alpha_{22}^T)\hat{q}_t - \frac{1}{q \Phi_1(q) y_2} \left[\Delta b_{1,t-1} - \beta \Delta b_{1,t} \right], \tag{58}$$

$$\widehat{rer}_t = -(1-\alpha_{11}^T - \alpha_{22}^T)\hat{q}_t, \tag{59}$$

where I have used the fact that in steady state $\frac{\Phi_1'(q)}{\Phi_1(q)} q = -(1-\alpha_{11}^T)$ and $\frac{\Phi_2'(q)}{\Phi_2(q)} q = (1-\alpha_{22}^T)$.

Using (57) and (58) in the approximation of the market clearing condition for good 2:

$$\frac{\partial z_2}{\partial q} q \hat{q}_t + \frac{\partial z_2}{\partial dW_1} \left[\Delta b_{1t-1} - \beta \Delta b_{1t} \right] = 0, \tag{60}$$

with the coefficients

$$\frac{\partial z_2}{\partial q} q = c_{12}(q) \left\{ \left(1-\varepsilon^T\right) \left(\frac{c_{11}(q)}{y_1^T} + \frac{c_{22}(q)}{y_2^T} \right) - 1 \right\}, \tag{61}$$

$$\frac{\partial z_2}{\partial dW_1} = -\frac{1}{q \Phi_1(q)} \left\{ \left(\frac{c_{11}(q)}{y_1^T} + \frac{c_{22}(q)}{y_2^T} \right) - 1 \right\}. \tag{62}$$

Note that in a steady state with balanced trade $\alpha_{ii} = \frac{c_{ii}(q)}{y_i^T}$. The slope of the excess demand function in steady state with respect to q is given by equation (61).

Aggregate consumption is determined by:

$$\hat{c}_{1,t}^T = -(1-\alpha_{11}^T)\hat{q}_t + \frac{1}{c_1(q)}(\Delta b_{1,t-1} - \beta \Delta b_{1,t}), \tag{63}$$

$$\hat{c}_{2,t}^T = (1-\alpha_{22}^T)\hat{q}_t - \frac{\Phi_2(q)}{q\Phi_1(q)}\frac{1}{c_2(q)}(\Delta b_{1,t-1} - \beta \Delta b_{1,t}), \tag{64}$$

while the risk sharing condition is given by:

$$\begin{aligned} \hat{c}_{1,t}^T - \hat{c}_{1,t+1}^T = \hat{c}_{2,t}^T - \hat{c}_{2,t+1}^T - (1-\alpha_{11}^T - \alpha_{22}^T)(\hat{q}_t - \hat{q}_{t+1}) \\ + \frac{\gamma}{\beta\Phi_1(q)}\left[1 + \frac{1}{q}\right]\Delta b_{1,t}. \end{aligned} \tag{65}$$

Thus, the dynamic system under portfolio costs can be written as:

$$\begin{pmatrix} -\frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial q}q}\beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta b_{1,t+1} \\ \Delta b_{1,t} \end{pmatrix} + \begin{pmatrix} \left(1 + \beta + \beta\frac{\frac{\partial z_2}{\partial q}q}{\overline{dz_2}}\check{\gamma}\right)\frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial q}q} & -\frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial q}q} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Delta b_{1,t} \\ \Delta b_{1,t-1} \end{pmatrix} = 0, \tag{66}$$

with the coefficients

$$\begin{aligned} \overline{dz_2} &= -\frac{1}{q\Phi_1(q)}\left\{\left(\frac{c_{11}(q)}{y_1^T} + \frac{c_{22}(q)}{y_2^T}\right) - 1\right\}^2 \\ &\quad -\frac{1}{q\Phi_1(q)}\left\{2 - \left(\frac{c_{11}(q)}{y_1^T} + \frac{c_{22}(q)}{y_2^T}\right)\right\}\left\{\frac{c_{11}(q)}{y_1^T} + \frac{c_{22}(q)}{y_2^T}\right\}\varepsilon^T < 0, \end{aligned} \tag{67}$$

$$\check{\gamma} = \frac{\gamma}{\beta^2\Phi_1(q)}\left[1 + \frac{1}{q}\right] > 0. \tag{68}$$

Hence, the characteristic equation associated with the dynamic system (66) under the portfolio cost approach satisfies:

$$\lambda^2 - \lambda\left(\frac{1+\beta}{\beta} + \frac{\frac{\partial z_2}{\partial q}q}{\overline{dz_2}}\check{\gamma}\right) + \frac{1}{\beta} = 0. \tag{69}$$

Alternatively, assume that households have Uzawa-type preferences. The time discount factor β^j is replaced by $\theta_{i,t+1+j} = \beta_i(c_{i,t+j}^T)\theta_{i,t+j}$ with $\beta_i(c_{i,t+j}^T) = \left(1 + \psi_i c_{i,t+j}^T\right)^{-1}$. ψ_i is chosen such that $\beta_i(c_i^T) = \beta$ in steady state. Absent internalization, the risk sharing condition implies:

$$\beta\hat{c}_{1,t}^T - \hat{c}_{1,t+1}^T = \beta\hat{c}_{2,t}^T - \hat{c}_{2,t+1}^T - (1-\alpha_{11}^T - \alpha_{22}^T)(\hat{q}_t - \hat{q}_{t+1}). \tag{70}$$

Hence, the characteristic equation associated with the dynamic system under endogenous discounting without internalization satisfies:

$$\lambda^2 - \left(\frac{1+\beta}{\beta} + \frac{\widetilde{dz_2}}{\overline{dz_2}}\right)\lambda + \left(1 + \frac{\widetilde{dz_2}}{\overline{dz_2}}\right)\frac{1}{\beta} = 0, \tag{71}$$

with

$$\widetilde{dz_2} = -\frac{1}{q\Phi_1(q)}\left[\frac{\beta_1'(q)}{\beta_1(q)}\left(1-\frac{c_{21}(q)}{y_1^T}\right)+\frac{\beta_2'(q)}{\beta_2(q)}\left(1-\frac{c_{22}(q)}{y_2^T}\right)\right]\varepsilon^T\left\{\frac{c_{11}(q)}{y_1^T}+\frac{c_{22}(q)}{y_2^T}\right\}>0, \quad (72)$$

as the discount factors are assumed to be decreasing in the level of consumption.[12]

B Appendix: Overlapping generations model

This appendix merges the framework with overlapping generations of infinitely lived households by Weil (1989) and Ghironi (2006) with the model presented in the main text. Absent Ricardian equivalence, the steady state level of the net foreign asset position can be shown to be unique. Furthermore, the local dynamics of the net foreign asset position are stationary. More details on the overlapping generations framework are provided in the Technical Appendix accompanying this paper.

Each household consumes, supplies labor, and holds financial assets. Households are born on different dates owning no assets, but they own the present discounted value of their labor income. The number of households in country i $N_{i,t}$ grows over time at the exogenous rate n_i, i.e., $N_{i,t+1} = (1+n_i)\,N_{i,t}$. The utility function of each household is assumed to be of the Cobb-Douglas type. Capital is no longer accumulated by households directly, but through capital producers. Households can purchase shares in these and all other firms that produce in their country of residence.

B.1 Individual Households

Consumers have identical preferences over the real consumption index and leisure. At time t_0, the representative consumer in country i born in period $v \in (-\infty, t_0)$ maximizes the intertemporal utility function:

$$U_{t_0}^v = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\frac{\left[\left(c_{i,t}^v\right)^{\xi_i}\left(1-l_{i,t}^v\right)^{1-\xi_i}\right]^{1-\sigma_i}-1}{1-\sigma_i} \quad (73)$$

s.t.

$$P_{i,t}^C c_{i,t}^v \le P_{i,t}w_{i,t}l_{i,t}^v + b_{i,t-1}^v - Q_{i,t}b_{i,t}^v + \left(Q_{i,t}^{\varpi}+P_{i,t}d_{i,t}\right)\varpi_{i,t-1}^v - Q_{i,t}^{\varpi}\varpi_{i,t}^v. \quad (74)$$

The final consumption of a representative household of country i born in period v at time t is denoted by $c_{i,t}^v$. The production of the final consumption good follows the same process as outlined in the main text. $l_{i,t}^v$ is the labor supply of such a household. The bond holdings are denoted by $b_{i,t}^v$. $\varpi_{i,t}^v$ are the household's holdings of the domestic stock, that sells at the price $Q_{i,t}^{\varpi}$ and pays the dividend $P_{i,t}d_{i,t}$. The supply of the stock is normalized to 1.

[12]If the discount factors were increasing in the level of consumption, $\widetilde{dz_2}$ would be negative and the dynamics around any deterministic steady state would be explosive.

B.2 Firms

While I assume that the production structure of the economy is unchanged relative to the main text, capital is held by capital producers. These firms buy the investment good, augment the existing capital stock, and rent out capital to the producers of traded and non-traded goods. They also pay a dividend to the stockholders. The optimization problem of the capital producers is given by:

$$\begin{aligned}
&\max_{k_{i,t},x_{i,t}} \sum_{t=t_0}^{\infty} Q_{i,t_0|t}(1+n_i)^t \left(P_{i,t} r_{i,t} k_{i,t-1} - P^I_{i,t} x_{i,t}\right) \\
&s.t. \\
&(1+n_i)k_{i,t} \leq (1-\delta)\, k_{i,t-1} + x_{i,t}. \qquad (75)
\end{aligned}$$

$Q_{i,t_0|t}$ is the pricing kernel used by the capital producers to discount future profits. The capital used in time t production, $k_{i,t-1}$, and investment, $x_{i,t}$, are expressed as per capita averages. The average per capital dividend payment is denoted by $P_{i,t}d_{i,t}$ and satisfies $P_{i,t}d_{i,t} = P_{i,t}r_{i,t}k_{i,t-1} - P^I_{i,t}x_{i,t}$.

Closing large open economy models: Technical Appendix

Martin Bodenstein**
Federal Reserve Board

Abstract

This Technical Appendix describes the details of the following topics in the paper "Closing large open economy models:" the simple example of Section 2, the linearized model equations of the model in the main text, the overlapping generations framework, and the model with real and nominal rigidities.

A Appendix: Details on the simple model

This appendix describes the equations underlying the discussion in Section 2. The first order conditions of the households imply:

$$c_{12,t} = \frac{\alpha_{12}^T q_t^{-\varepsilon^T}}{\alpha_{11}^T + \alpha_{12}^T q_t^{1-\varepsilon^T}} \left[y_{1,t}^T + \frac{b_{1,t-1} - Q_{1,t} b_{1,t}}{\Phi_1(q_t)} \right], \tag{1}$$

$$c_{22,t} = \frac{\alpha_{22}^T}{\alpha_{21}^T \left(\frac{1}{q_t}\right)^{1-\varepsilon^T} + \alpha_{22}^T} \left[y_{2,t}^T + \frac{b_{2,t-1} - Q_{2,t} b_{2,t}}{\Phi_1(q_t) q_t} \right], \tag{2}$$

$$c_{1,t}^T = \Phi_1(q_t) y_{1,t}^T + b_{1,t-1} - Q_{1,t} b_{1,t}, \tag{3}$$

$$c_{2,t}^T = \Phi_2(q_t) y_{2,t}^T + \frac{\Phi_2(q_t)}{q_t \Phi_1(q_t)} \left(b_{2,t-1} - Q_{2,t} b_{2,t} \right), \tag{4}$$

where

$$\Phi_1(q_t) \equiv \frac{P_{1,t}}{P_{1,t} c_{11,t} + P_{2,t} c_{12,t}} = \left[\alpha_{11}^T \left(\frac{1}{q_t} \right)^{1-\varepsilon^T} + \alpha_{12}^T \right]^{-\frac{1}{1-\varepsilon^T}} \frac{1}{q_t}, \tag{5}$$

$$\Phi_2(q_t) \equiv \frac{P_{2,t}}{P_{1,t} c_{21,t} + P_{2,t} c_{22,t}} = \left[\alpha_{21}^T \left(\frac{1}{q_t} \right)^{1-\varepsilon^T} + \alpha_{22}^T \right]^{-\frac{1}{1-\varepsilon^T}}, \tag{6}$$

$$rer_t = q_t \frac{\Phi_1(q_t)}{\Phi_2(q_t)}. \tag{7}$$

The price of the aggregate consumption good in country 1 is taken to be the numéraire. Log-linearizing these equations around a steady state with zero net foreign assets and setting the exogenous variables $y_{1,t}^T$ and $y_{2,t}^T$ to be constant ("$\hat{x}$" for log-linearized and "Δx" for linearized variables x). Variables without time index indicate steady state values:

$$\hat{c}_{12,t} = -\left\{ (1-\varepsilon^T)(1-\alpha_{11}^T) + \varepsilon^T \right\} \hat{q}_t + \frac{1}{y_1 \Phi_1(q)} \left[\Delta b_{1,t-1} - \beta \Delta b_{1,t} \right], \tag{8}$$

$$\hat{c}_{22,t} = (1-\varepsilon^T)(1-\alpha_{22}^T)\hat{q}_t - \frac{1}{q \Phi_1(q) y_2} \left[\Delta b_{1,t-1} - \beta \Delta b_{1,t} \right], \tag{9}$$

$$\widehat{rer}_t = -(1-\alpha_{11}^T - \alpha_{22}^T)\hat{q}_t, \tag{10}$$

where I have used the fact that in steady state $\frac{\Phi_1'(q)}{\Phi_1(q)} q = -(1-\alpha_{11}^T)$ and $\frac{\Phi_2'(q)}{\Phi_2(q)} q = (1-\alpha_{22}^T)$.

Using (8) and (9) in the approximation of the market clearing condition for good 2:

$$\frac{\partial z_2}{\partial q} q \hat{q}_t + \frac{\partial z_2}{\partial dW_1} \left[\Delta b_{1t-1} - \beta \Delta b_{1t} \right] = 0, \tag{11}$$

with the coefficients

$$\begin{aligned}
\frac{\partial z_2}{\partial q} q &= c_{12}(q) \left\{ \left(1-\varepsilon^T\right) \left(\frac{c_{11}(q)}{y_1^T} + \frac{c_{22}(q)}{y_2^T} \right) - 1 \right\}, && (12) \\
\frac{\partial z_2}{\partial dW_1} &= -\frac{1}{q\Phi_1(q)} \left\{ \left(\frac{c_{11}(q)}{y_1^T} + \frac{c_{22}(q)}{y_2^T} \right) - 1 \right\}. && (13)
\end{aligned}$$

Note that in a steady state with balanced trade $\alpha_{ii} = \frac{c_{ii}(q)}{y_i^T}$. The slope of the excess demand function in steady state with respect to q is given by equation (12).

Aggregate consumption is determined by:

$$\begin{aligned}
\hat{c}_{1,t}^T &= -(1-\alpha_{11}^T)\hat{q}_t + \frac{1}{c_1(q)}(\Delta b_{1,t-1} - \beta \Delta b_{1,t}), && (14) \\
\hat{c}_{2,t}^T &= (1-\alpha_{22}^T)\hat{q}_t - \frac{\Phi_2(q)}{q\Phi_1(q)} \frac{1}{c_2(q)}(\Delta b_{1,t-1} - \beta \Delta b_{1,t}), && (15)
\end{aligned}$$

while the risk sharing condition is given by:

$$\begin{aligned}
\hat{c}_{1,t}^T - \hat{c}_{1,t+1}^T = \hat{c}_{2,t}^T - \hat{c}_{2,t+1}^T - (1-\alpha_{11}^T - \alpha_{22}^T)(\hat{q}_t - \hat{q}_{t+1}) \\
+ \frac{\gamma}{\beta \Phi_1(q)} \left[1 + \frac{1}{q} \right] \Delta b_{1,t}. && (16)
\end{aligned}$$

Thus, the dynamic system under portfolio costs can be written as:

$$\begin{pmatrix} -\frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial q} q}\beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta b_{1,t+1} \\ \Delta b_{1,t} \end{pmatrix} + \begin{pmatrix} \left(1+\beta+\beta \frac{\frac{\partial z_2}{\partial q} q}{\overline{dz_2}} \check{\gamma}\right) \frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial q} q} & -\frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial q} q} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Delta b_{1,t} \\ \Delta b_{1,t-1} \end{pmatrix} = 0, \tag{17}$$

with the coefficients

$$\begin{aligned}
\overline{dz_2} &= -\frac{1}{q\Phi_1(q)} \left\{ \left(\frac{c_{11}(q)}{y_1^T} + \frac{c_{22}(q)}{y_2^T} \right) - 1 \right\}^2 \\
&\quad - \frac{1}{q\Phi_1(q)} \left\{ 2 - \left(\frac{c_{11}(q)}{y_1^T} + \frac{c_{22}(q)}{y_2^T} \right) \right\} \left\{ \frac{c_{11}(q)}{y_1^T} + \frac{c_{22}(q)}{y_2^T} \right\} \varepsilon^T < 0, && (18) \\
\check{\gamma} &= \frac{\gamma}{\beta^2 \Phi_1(q)} \left[1 + \frac{1}{q} \right] > 0. && (19)
\end{aligned}$$

Hence, the characteristic equation associated with the dynamic system (17) under the portfolio cost approach satisfies:

$$\lambda^2 - \lambda \left(\frac{1+\beta}{\beta} + \frac{\frac{\partial z_2}{\partial q} q}{\overline{dz_2}} \check{\gamma} \right) + \frac{1}{\beta} = 0. \tag{20}$$

Alternatively, assume that households have Uzawa-type preferences. The time discount factor β^j is replaced by $\theta_{i,t+1+j} = \beta_i(c^T_{i,t+j})\theta_{i,t+j}$ with $\beta_i(c^T_{i,t+j}) = \left(1+\psi_i c^T_{i,t+j}\right)^{-1}$. ψ_i is chosen such that $\beta_i(c^T_i) = \beta$ in steady state. Absent internalization, the risk sharing condition implies:

$$\beta\hat{c}^T_{1,t} - \hat{c}^T_{1,t+1} = \beta\hat{c}^T_{2,t} - \hat{c}^T_{2,t+1} - (1-\alpha^T_{11} - \alpha^T_{22})(\hat{q}_t - \hat{q}_{t+1}). \tag{21}$$

Hence, the characteristic equation associated with the dynamic system under endogenous discounting without internalization satisfies:

$$\lambda^2 - \left(\frac{1+\beta}{\beta} + \frac{\widetilde{dz_2}}{\overline{\overline{dz_2}}}\right)\lambda + \left(1 + \frac{\widetilde{dz_2}}{\overline{\overline{dz_2}}}\right)\frac{1}{\beta} = 0, \tag{22}$$

with

$$\widetilde{dz_2} = -\frac{1}{q\Phi_1(q)}\left[\frac{\beta_1'(q)}{\beta_1(q)}\left(1-\frac{c_{21}(q)}{y^T_1}\right) + \frac{\beta_2'(q)}{\beta_2(q)}\left(1-\frac{c_{22}(q)}{y^T_2}\right)\right]\varepsilon^T\left\{\frac{c_{11}(q)}{y^T_1} + \frac{c_{22}(q)}{y^T_2}\right\} > 0, \tag{23}$$

as the discount factors are assumed to be decreasing in the level of consumption.[1]

B Appendix: Linearized model

This appendix describes the nonlinear first order conditions together with their linear approximations. $\hat{x}$ denotes the log-linear approximation of variable x from its steady state value, Δx denotes its deviation in absolute value:

1. choice of leisure in country i:

   - with additive separable preferences

     $$\chi_0 l^{\chi}_{i,t} = \Phi_{i,t} w_{i,t} c^{-\sigma}_{i,t}$$

     is approximated by

     $$\chi\hat{l}_{i,t} = \hat{\Phi}_{i,t} + \hat{w}_{i,t} - \sigma\hat{c}_{i,t}$$

   - with Cobb-Douglas preferences

     $$(1-\xi)\frac{1}{(1-l_{i,t})} = \Phi_{i,t} w_{i,t} \xi \frac{1}{c_{i,t}}$$

[1] If the discount factors were increasing in the level of consumption, $\widetilde{dz_2}$ would be negative and the dynamics around any deterministic steady state would be explosive.

is approximated by

$$\frac{l_i}{1-l_i}\hat{l}_{i,t} = \hat{\Phi}_{i,t} + \hat{w}_{i,t} - \hat{c}_{i,t}$$

2. choice of capital in country i:

- with additive separable preferences

$$\beta \left(\frac{c_{i,t+1}}{c_{i,t}}\right)^{-\sigma} \frac{\Phi_{i,t+1}}{\Phi_{i,t}} \frac{\Pi_{i,t}}{\Pi_{i,t+1}} \left[\Pi_{i,t+1} r_{i,t+1} + (1-\delta)\right] = 1$$

is approximated by

$$\begin{aligned}
&\beta \Pi_i r_i \left[\hat{\Pi}_{i,t+1} + \hat{r}_{i,t+1}\right] - \sigma \left(\hat{c}_{i,t+1} - \hat{c}_{i,t}\right) \\
&+ \left(\hat{\Phi}_{i,t+1} - \hat{\Phi}_{i,t}\right) - \left(\hat{\Pi}_{i,t+1} - \hat{\Pi}_{i,t}\right) = 0
\end{aligned}$$

which is augmented by $\hat{\beta}_{i,t}$ on the left side under endogenous discounting

- with Cobb-Douglas preferences

$$\beta \left(\frac{c_{i,t+1}^{\xi} \left(1-l_{i,t+1}\right)^{1-\xi}}{c_{i,t}^{\xi} \left(1-l_{i,t}\right)^{1-\xi}}\right)^{1-\sigma} \frac{c_{i,t}}{c_{i,t+1}} \frac{\Phi_{i,t+1}}{\Phi_{i,t}} \frac{\Pi_{i,t}}{\Pi_{i,t+1}} \left[\Pi_{i,t+1} r_{i,t+1} + (1-\delta)\right] = 1$$

is approximated by

$$\begin{aligned}
&\left\{(1-\sigma)\xi - 1\right\}\left(\hat{c}_{i,t+1} - \hat{c}_{i,t}\right) - (1-\sigma)(1-\xi)\, l_i(\hat{l}_{i,t+1} - \hat{l}_{i,t}) \\
&+ \hat{\Phi}_{i,t+1} - \hat{\Phi}_{i,t} + \hat{\Pi}_{i,t} - \hat{\Pi}_{i,t+1} + \beta \Pi_i r_i \left(\hat{\Pi}_{i,t+1} + \hat{r}_{i,t+1}\right) = 0
\end{aligned}$$

which is augmented by $\hat{\beta}_{i,t}$ on the left side under endogenous discounting

3. capital accumulation in country i:

$$k_{i,t} \leq (1-\delta)\, k_{i,t-1} + x_{i,t}$$

is approximated by

$$\hat{k}_{i,t} \leq (1-\delta)\, \hat{k}_{i,t-1} + \delta \hat{x}_{i,t}$$

4. consumer budget constraint

- for country 1:

$$c_{1,t} = \Phi_{1,t}\left(y_{1,t}^T + q_{1,t}^N y_{1,t}^N\right) - \frac{\Phi_{1,t}}{\Pi_{1,t}} x_{1,t} + dW_{1,t}$$

is approximated by

$$\begin{aligned}
c_1 \hat{c}_{1,t} = {} & \left(y_1^T + q_1^N y_1^N\right) \Phi_1 \hat{\Phi}_{1,t} + \Phi_1 \left(y_1^T \hat{y}_{1,t}^T + q_1^N y_1^N \hat{y}_{1,t}^N + q_1^N y_1^N \hat{q}_{1,t}^N\right) \\
& - \frac{\Phi_1}{\Pi_1} x_1 \left(\hat{\Phi}_{1,t} - \hat{\Pi}_{1,t}\right) - \frac{\Phi_1}{\Pi_1} x_1 \hat{x}_{1,t} + \Delta dW_{1,t}
\end{aligned}$$

- for country 2:

$$c_{2,t} = \Phi_{2,t}\left(y_{2,t}^T + q_{2,t}^N y_{2,t}^N\right) - \frac{\Phi_{2,t}}{\Pi_{2,t}} x_{2,t} - \frac{1}{rer_t}\frac{1}{\nu_2} dW_{1,t}$$

is approximated by

$$\begin{aligned} c_2 \hat{c}_{2,t} &= \left(y_2^T + q_2^N y_2^N\right)\Phi_2\hat{\Phi}_{2,t} + \Phi_2\left(y_2^T\hat{y}_{2,t}^T + q_2^N y_2^N \hat{y}_{2,t}^N + q_2^N y_2^N \hat{q}_{2,t}^N\right) \\ &\quad - \frac{\Phi_2}{\Pi_2} x_2\left(\hat{\Phi}_{2,t} - \hat{\Pi}_{2,t}\right) - \frac{\Phi_2}{\Pi_2} x_2 \hat{x}_{2,t} - \frac{1}{rer}\frac{1}{\nu_2}\Delta dW_{1,t} \end{aligned}$$

and under complete markets, the two budget constraints are aggregated to deliver

$$\begin{aligned} c_{1,t} + \nu_2 rer_t c_{2,t} &= \Phi_{1,t}\left(w_{1,t} l_{1,t} + r_{1,t} k_{1,t-1}\right) - \frac{\Phi_{1,t}}{\Pi_{1,t}} x_{1,t} \\ &\quad + \nu_2 rer_t\left(\Phi_{2,t}\left(w_{2,t} l_{2,t} + r_{2,t} k_{2,t-1}\right) - \frac{\Phi_{2,t}}{\Pi_{2,t}} x_{2,t}\right) \end{aligned}$$

5. risk sharing condition with additive separable preferences

- with adjustment costs:

$$\beta_1\left(\frac{U_{c1,t+1}}{U_{c1,t}}\right) - \beta_2\left(\frac{U_{c2,t+1}}{U_{c2,t}}\right)\frac{rer_t}{rer_{t+1}} = \Gamma'\left(\frac{b_{1,t}}{P_{1,t}}\right) - \Gamma'\left(\frac{b_{2,t}}{P_{2,t}}\right)$$

under additive separable preferences is approximated by

$$-\sigma_1\left(\hat{c}_{1,t+1} - \hat{c}_{1,t}\right) + \sigma_2\left(\hat{c}_{2,t+1} - \hat{c}_{2,t}\right) - \widehat{rer}_t + \widehat{rer}_{t+1} = -\phi_b\left(\frac{1}{\Phi_1} + \frac{1}{rer\Phi_2}\right)\Delta b_t$$

- with debt-elastic interest rate:

$$\frac{\mathbb{E}_t \frac{U_{c2,t+1}}{U_{c2,t}}\frac{rer_t}{rer_{t+1}}}{\mathbb{E}_t \frac{U_{c1,t+1}}{U_{c1,t}}} = \Psi\left(b_{1,t+1} - \bar{b}\right)$$

is approximated by

$$-\sigma_1\left(\hat{c}_{1,t+1} - \hat{c}_{1,t}\right) + \sigma_2\left(\hat{c}_{2,t+1} - \hat{c}_{2,t}\right) - \widehat{rer}_t + \widehat{rer}_{t+1} = -\Psi'(0)\,\Delta b_{1,t+1}$$

- with endogenous discounting, but no internalization

$$\beta_{1,t}\left(\frac{U_{c1,t+1}}{U_{c1,t}}\right) - \beta_{2,t}\left(\frac{U_{c2,t+1}}{U_{c2,t}}\right)\frac{rer_t}{rer_{t+1}} = 0$$

under additive separable preferences is approximated by

$$-\sigma_1\left(\hat{c}_{1,t+1} - \hat{c}_{1,t}\right) + \sigma_2\left(\hat{c}_{2,t+1} - \hat{c}_{2,t}\right) - \widehat{rer}_t + \widehat{rer}_{t+1} = -\hat{\beta}_{1,t} + \hat{\beta}_{2,t}$$

- with endogenous discounting and internalization:

$$\beta_{1,t}\frac{U_{c1,t+1} - \eta_{1,t+1}\beta'_{1,t+1}}{U_{c1,t} - \eta_{1,t}\beta'_{1,t}} = \beta_{2,t}\frac{U_{c2,t+1} - \eta_{2,t+1}\beta'_{2,t+1}}{U_{c2,t} - \eta_{2,t}\beta'_{2,t}}\frac{rer_t}{rer_{t+1}}$$

$$\eta_{i,t} = E_t\left(-U_{i,t+1} + \beta_{i,t+1}\eta_{i,t+1}\right)$$

is approximated by

$$\begin{aligned}
& \beta_1' U_{c1} c_1 \hat{c}_{1,t} + \beta_1' U_{l1} l_1 \hat{l}_{1,t} - \beta_1 \frac{\eta_1 \beta_1'}{U_{c1} - \eta_1 \beta_1'} \left(\hat{\eta}_{1,t+1} - \hat{\eta}_{1,t} \right) \\
& + \beta_1 \frac{U_{cc1} - \eta_1 \beta_1'' U_{c1}}{U_{c1} - \eta_1 \beta_1'} c_1 \left(\hat{c}_{1,t+1} - \hat{c}_{1,t} \right) + \beta_1 \frac{U_{cl1} - \eta_1 \beta_1'' U_{l1}}{U_{c1} - \eta_1 \beta_1'} l_1 \left(\hat{l}_{1,t+1} - \hat{l}_{1,t} \right) \\
= \; & \beta_2' U_{c2} c_2 \hat{c}_{2,t} + \beta_2' U_{l2} l_2 \hat{l}_{2,t} - \beta_2 \frac{\eta_2 \beta_2'}{U_{c2} - \eta_2 \beta_2'} \left(\hat{\eta}_{2,t+1} - \hat{\eta}_{2,t} \right) \\
& + \beta_2 \frac{U_{cc2} - \eta_2 \beta_2'' U_{c2}}{U_{c2} - \eta_2 \beta_2'} c_2 \left(\hat{c}_{2,t+1} - \hat{c}_{2,t} \right) + \beta_2 \frac{U_{cl2} - \eta_2 \beta_2'' U_{l2}}{U_{c2} - \eta_2 \beta_2'} l_2 \left(\hat{l}_{2,t+1} - \hat{l}_{2,t} \right) \\
& - \widehat{rer}_t + \widehat{rer}_{t+1}
\end{aligned}$$

and

$$\hat{\eta}_{i,t} = -\frac{1 - \eta_i \beta_i'}{\eta_i} \left(U_{ci} c_i \hat{c}_{i,t+1} + U_{li} l_i \hat{l}_{i,t+1} \right) + \beta_i \hat{\eta}_{i,t+1}$$

6. endogenous discount factors

- with additive separable preferences:

$$\beta_{i,t} = 1 - \psi_i \exp \left(\frac{c_{i,t}^{1-\sigma_i}}{1 - \sigma_i} - \chi_0 \frac{l_{i,t}^{1+\chi_i}}{1 + \chi_i} \right)$$

is approximated by

$$\hat{\beta}_{i,t} = \frac{\beta - 1}{\beta} \left[c_i^{1-\sigma_i} \hat{c}_{i,t} - \chi_0 l_i^{1+\chi_i} \hat{l}_{i,t} \right]$$

- with Cobb Douglas preferences

$$\beta_{i,t} = \left(1 + \psi \left[c_{i,t}^{\xi} \left(1 - l_{i,t} \right)^{1-\xi} \right] \right)^{-1}$$

is approximated by

$$\hat{\beta}_{i,t} = -\left(1 - \beta_i \right) \xi \hat{c}_{i,t} + \left(1 - \beta_i \right) \frac{l_i}{1 - l_i} \left(1 - \xi \right) \hat{l}_{i,t}$$

7. wealth evolution

$$dW_{1,t} = b_{1,t-1} - Q_{1,t} b_{1,t}$$

where $Q_{1,t} = \beta_{1,t} \frac{U_{c1,t+1}}{U_{c1,t}}$ implies the approximation

$$\Delta dW_{1,t} = \Delta b_{1,t-1} - \beta \Delta b_{1,t}$$

8. investment aggregate for country i

$$x_{i,t} = \left[\left(\alpha_{i1}^I \right)^{\frac{1}{\varepsilon_i^I}} \left(x_{i1,t} \right)^{\frac{\varepsilon_i^I - 1}{\varepsilon_i^I}} + \left(\alpha_{i2}^I \right)^{\frac{1}{\varepsilon_i^I}} \left(x_{i2,t} \right)^{\frac{\varepsilon_i^I - 1}{\varepsilon_i^I}} \right]^{\frac{\varepsilon_i^I}{\varepsilon_i^I - 1}}$$

is approximated by

$$\hat{x}_{i,t} = \alpha_{i1}^{I}\left(\frac{x_{i1}}{\alpha_{i1}^{I}x_i}\right)^{\frac{\varepsilon_i^I-1}{\varepsilon_i^I}}\hat{x}_{i1,t}+\alpha_{i2}^{I}\left(\frac{x_{i2}}{\alpha_{i2}^{I}x_i}\right)^{\frac{\varepsilon_i^I-1}{\varepsilon_i^I}}\hat{x}_{i2,t}$$

9. investment optimal choices

   - for country 1:

     $$x_{12,t} = \frac{\alpha_{12}^{I}}{\alpha_{11}^{I}}\left(\frac{1}{q_t}\right)^{\varepsilon_1^I}x_{11,t}$$

     is approximated by

     $$\hat{x}_{12,t} = \hat{x}_{11,t}-\varepsilon_1^T\hat{q}_t$$

   - for country 2:

     $$x_{22,t} = \frac{\alpha_{22}^{I}}{\alpha_{21}^{I}}\left(\frac{1}{q_t}\right)^{\varepsilon_2^I}x_{21,t}$$

     is approximated by

     $$\hat{x}_{22,t} = \hat{x}_{21,t}-\varepsilon_2^T\hat{q}_t$$

10. consumption aggregate over traded goods for country i:

$$c_{i,t}^{T} = \left[\left(\alpha_{i1}^{T}\right)^{\frac{1}{\varepsilon_i^T}}\left(c_{i1,t}\right)^{\frac{\varepsilon_i^T-1}{\varepsilon_i^T}}+\left(\alpha_{i2}^{T}\right)^{\frac{1}{\varepsilon_i^T}}\left(c_{i2,t}\right)^{\frac{\varepsilon_i^T-1}{\varepsilon_i^T}}\right]^{\frac{\varepsilon_i^T}{\varepsilon_i^T-1}}$$

is approximated by

$$\hat{c}_{i,t}^{T} = \alpha_{i1}^{T}\left(\frac{c_{i1}}{\alpha_{i1}^{T}c_i^T}\right)^{\frac{\varepsilon_i^T-1}{\varepsilon_i^T}}\hat{c}_{i1,t}+\alpha_{i2}^{T}\left(\frac{c_{i2}}{\alpha_{i2}^{T}c_i^T}\right)^{\frac{\varepsilon_i^T-1}{\varepsilon_i^T}}\hat{c}_{i2,t}$$

11. consumption aggregate over traded and non traded goods for country i

$$c_{i,t} = \left[\left(\alpha_{i1}^{N}\right)^{\frac{1}{\varepsilon_i^N}}\left(c_{i,t}^{T}\right)^{\frac{\varepsilon_i^N-1}{\varepsilon_i^N}}+\left(\alpha_{i2}^{N}\right)^{\frac{1}{\varepsilon_i^N}}\left(c_{i,t}^{N}\right)^{\frac{\varepsilon_i^N-1}{\varepsilon_i^N}}\right]^{\frac{\varepsilon_i^N}{\varepsilon_i^N-1}}$$

is approximated by

$$\hat{c}_{i,t} = \alpha_{i1}^{N}\left(\frac{c_i^T}{\alpha_{i1}^{N}c_i}\right)^{\frac{\varepsilon_i^N-1}{\varepsilon_i^N}}\hat{c}_{i,t}^{T}+\alpha_{i2}^{N}\left(\frac{c_i^N}{\alpha_{i2}^{N}c_i}\right)^{\frac{\varepsilon_i^N-1}{\varepsilon_i^N}}\hat{c}_{i,t}^{N}$$

12. consumption optimal choices non-traded goods

- for country 1:

$$c_{1,t}^N = c_{12,t}\frac{\alpha_{12}^N}{\alpha_{11}^N}\left[\left(\alpha_{11}^T\right)^{\frac{1}{\varepsilon_1^T}}\left(\frac{c_{11,t}}{c_{12,t}}\right)^{\frac{\varepsilon_1^T-1}{\varepsilon_1^T}}+\left(\alpha_{12}^T\right)^{\frac{1}{\varepsilon_1^T}}\right]^{\frac{\varepsilon_1^T-\varepsilon_1^N}{\varepsilon_1^T-1}}\left(\frac{q_{1,t}^N}{q_t+\eta q_{1,t}^N}\right)^{-\varepsilon_1^N}\left(\frac{1}{\alpha_{12}^T}\right)^{\frac{\varepsilon_1^N}{\varepsilon_1^T}}$$

is approximated by

$$\begin{aligned}\hat{c}_{1,t}^N &= \hat{c}_{12,t}+\frac{\varepsilon_1^T-\varepsilon_1^N}{\varepsilon_1^T}\frac{\left(\alpha_{11}^T\right)^{\frac{1}{\varepsilon_1^T}}\left(\frac{c_{11}}{c_{12}}\right)^{\frac{\varepsilon_1^T-1}{\varepsilon_1^T}}}{\left(\alpha_{11}^T\right)^{\frac{1}{\varepsilon_1^T}}\left(\frac{c_{11}}{c_{12}}\right)^{\frac{\varepsilon_1^T-1}{\varepsilon_1^T}}+\left(\alpha_{12}^T\right)^{\frac{1}{\varepsilon_1^T}}}\left(\hat{c}_{11,t}-\hat{c}_{12,t}\right)\\ &\quad -\varepsilon_1^N\frac{q}{q+\eta q_1^N}\left(\hat{q}_{1,t}^N-\hat{q}_t\right)\end{aligned}$$

- for country 2:

$$c_{2,t}^N = c_{21,t}\frac{\alpha_{22}^N}{\alpha_{21}^N}\left[\left(\alpha_{21}^T\right)^{\frac{1}{\varepsilon_2^T}}+\left(\alpha_{22}^T\right)^{\frac{1}{\varepsilon_2^T}}\left(\frac{c_{22,t}}{c_{21,t}}\right)^{\frac{\varepsilon_2^T-1}{\varepsilon_2^T}}\right]^{\frac{\varepsilon_2^T-\varepsilon_2^N}{\varepsilon_2^T-1}}\left(\frac{q_{2,t}^N}{\frac{1}{q_t}+\eta q_{2,t}^N}\right)^{-\varepsilon_2^N}\left(\frac{1}{\alpha_{21}^T}\right)^{\frac{\varepsilon_2^N}{\varepsilon_2^T}}$$

is approximated by

$$\begin{aligned}c_2^N\hat{c}_{2,t}^N &= \hat{c}_{21,t}+\frac{\varepsilon_2^T-\varepsilon_2^N}{\varepsilon_2^T}\frac{\left(\alpha_{22}^T\right)^{\frac{1}{\varepsilon_2^T}}\left(\frac{c_{22}}{c_{21}}\right)^{\frac{\varepsilon_2^T-1}{\varepsilon_2^T}}}{\left(\alpha_{21}^T\right)^{\frac{1}{\varepsilon_2^T}}+\left(\alpha_{22}^T\right)^{\frac{1}{\varepsilon_2^T}}\left(\frac{c_{22}}{c_{21}}\right)^{\frac{\varepsilon_2^T-1}{\varepsilon_2^T}}}\left(\hat{c}_{22,t}-\hat{c}_{21,t}\right)\\ &\quad -\varepsilon_2^N\frac{\frac{1}{q}}{\frac{1}{q}+\eta q_2^N}\left(\hat{q}_{2,t}^N+\hat{q}_t\right)\end{aligned}$$

13. consumption optimal choice traded goods

- for country 1:

$$\begin{aligned}c_{12,t} &= \frac{\alpha_{12}^T}{\alpha_{11}^T}\left(\frac{1+\eta q_{1,t}^N}{q_t+\eta q_{1,t}^N}\right)^{\varepsilon_1^T}c_{11,t}\\ \hat{c}_{12,t} &= \hat{c}_{11,t}+\left[\frac{1}{1+\eta q_1^N}-\frac{1}{q+\eta q_1^N}\right]\varepsilon_1^T\eta q_1^N\hat{q}_{1,t}^N-\frac{\varepsilon_1^T}{q+\eta q_1^N}q\hat{q}_t\end{aligned}$$

- for country 2:

$$\begin{aligned}c_{22,t} &= \frac{\alpha_{22}^T}{\alpha_{21}^T}\left(\frac{\frac{1}{q_t}+\eta q_{2,t}^N}{1+\eta q_{2,t}^N}\right)^{\varepsilon_2^T}c_{21,t}\\ \hat{c}_{22,t} &= \hat{c}_{21,t}+\left[\frac{1}{\frac{1}{q}+\eta q_2^N}-\frac{1}{1+\eta q_2^N}\right]\varepsilon_2^T\eta q_2^N\hat{q}_{2,t}^N-\frac{\varepsilon_2^T}{\frac{1}{q}+\eta q_2^N}\frac{1}{q}\hat{q}_t\end{aligned}$$

14. definition Φ_i

$$\Phi_{i,t} = \Phi_i\left(q_t, q_{i,t}^N\right)$$
$$\hat{\Phi}_{i,t} = \frac{\partial \Phi_i}{\partial q}\frac{q}{\Phi_i}\hat{q}_t + \frac{\partial \Phi_i}{\partial q_i^N}\frac{q_i^N}{\Phi_i}\hat{q}_{i,t}^N$$

- for country 1:
$$\frac{\partial \Phi_1}{\partial q}\frac{q}{\Phi_1} = -\frac{\frac{c_{12}}{c_1^N}q}{\left[(1+\eta q_1^N)\frac{c_{11}}{c_{12}} + (q+\eta q_1^N)\right]\frac{c_{12}}{c_1^N} + q_1^N} \text{ and } \frac{\partial \Phi_1}{\partial q_1^N}\frac{q_1^N}{\Phi_1} = \frac{\partial \Phi_1}{\partial q}\frac{q}{\Phi_1}\frac{y_1^N q_1^N}{c_{12}q}$$

- for country 2:
$$\frac{\partial \Phi_2}{\partial q}\frac{q}{\Phi_2} = \frac{\frac{c_{21}}{c_2^N}\frac{1}{q}}{\left[\left(\frac{1}{q}+\eta q_2^N\right)\frac{c_{21}}{c_{22}} + (1+\eta q_2^N)\right]\frac{c_{22}}{c_2^N} + q_2^N} \text{ and } \frac{\partial \Phi_2}{\partial q_2^N}\frac{q_2^N}{\Phi_2} = -\frac{y_2^N q_2^N}{c_{21}\frac{1}{q}}\frac{\partial \Phi_2}{\partial q}\frac{q}{\Phi_2}$$

15. definition Π_i

$$\Pi_{i,t} = \Pi_i(q_t)$$
$$\hat{\Pi}_{i,t} = \frac{\partial \Pi_i}{\partial q}\frac{q}{\Pi_i}\hat{q}_t$$

- for country 1:
$$\frac{\partial \Pi_1}{\partial q}\frac{q}{\Pi_1} = -\frac{q}{\frac{i_{11}}{i_{12}} + q}$$

- for country 2:
$$\frac{\partial \Pi_2}{\partial q}\frac{q}{\Pi_2} = \frac{\frac{i_{21}}{i_{22}}}{\frac{i_{21}}{i_{22}} + q}$$

16. real exchange rate

$$rer_t = \frac{\Phi_{1,t}}{\Phi_{2,t}}q_t$$

is approximated by

$$\widehat{rer}_t = \hat{\Phi}_{1,t} - \hat{\Phi}_{2,t} + \hat{q}_t$$

17. Cobb-Douglas production function with traded goods for country i:

$$y_{i,t}^T = \left(\frac{A_{i,t}^T l_{i,t}^T}{\omega_{li}^T}\right)^{\omega_{li}^T}\left(\frac{k_{i,t}^T}{\omega_{ki}^T}\right)^{\omega_{ki}^T}$$

is approximated by

$$\hat{y}^T_{i,t} = \omega^T_{li}\hat{A}^T_{i,t} + \omega^T_{li}\hat{l}^T_{i,t} + \omega^T_{ki}\hat{k}^T_{i,t}$$

18. optimal labor input choices traded goods for country i:

$$w_{i,t} = \frac{\omega^T_{li}y^T_{i,t}}{l^T_{i,t}}$$

is approximated by

$$\hat{w}_{i,t} = \hat{y}^T_{i,t} - \hat{l}^T_{i,t}$$

19. optimal capital input choices traded goods for country i:

$$r_{i,t} = \frac{\omega^T_{kli}y^T_{i,t}}{k^T_{i,t}}$$

is approximated by

$$\hat{r}_{i,t} = \hat{y}^T_{i,t} - \hat{k}^T_{i,t}$$

20. Cobb-Douglas production function with non-traded goods for country i:

$$y^N_{i,t} = \left(\frac{A^N_{i,t}l^N_{i,t}}{\omega^N_{li}}\right)^{\omega^N_{li}} \left(\frac{k^N_{i,t}}{\omega^N_{ki}}\right)^{\omega^N_{ki}}$$

is approximated by

$$\hat{y}^N_{i,t} = \omega^N_{li}\hat{A}^N_{i,t} + \omega^N_{li}\hat{l}^N_{i,t} + \omega^N_{ki}\hat{k}^N_{i,t}$$

21. optimal labor input choices non-traded goods for country i:

$$w_{i,t} = \frac{P^N_{i,t}}{P_{i,t}}\left(\frac{\omega^N_{li}y^N_{i,t}}{l^N_{i,t}}\right)$$

is approximated by

$$\hat{w}_{i,t} = \hat{q}^N_{i,t} + \hat{y}^N_{i,t} - \hat{l}^N_{i,t}$$

22. optimal capital input choices non-traded goods for country i:

$$r_{i,t} = \frac{P^N_{i,t}}{P_{i,t}}\left(\frac{\omega^N_{kli}y^N_{i,t}}{k^N_{i,t}}\right)$$

is approximated by

$$\hat{r}_{i,t} = \hat{q}^N_{i,t} + \hat{y}^N_{i,t} - \hat{k}^N_{i,t}$$

23. labor market clearing for country i:

$$l^T_{i,t} + l^N_{i,t} = l_{i,t}$$

is approximated by

$$\frac{l^T_i}{l_i}\hat{l}^T_{i,t} + \frac{l^N_i}{l_i}\hat{l}^N_{i,t} = \hat{l}_{i,t}$$

24. capital market clearing for country i:

$$k^T_{i,t} + k^N_{i,t} = k_{i,t-1}$$

is approximated by

$$\frac{k^T_i}{k_i}\hat{k}^T_{i,t} + \frac{k^N_i}{k_i}\hat{k}^N_{i,t} = \hat{k}_{i,t-1}$$

25. goods market clearing non-traded goods for country i:

$$y^N_{i,t} = c^N_{i,t} + \eta\left(c_{i1,t} + c_{i2,t}\right)$$

is approximated by

$$\hat{y}^N_{i,t} = \frac{c^N_i}{y^N_i}\hat{c}^N_{i,t} + \frac{\eta c_{i1}}{y^N_i}\hat{c}_{i1,t} + \frac{\eta c_{i2}}{y^N_i}\hat{c}_{i2,t}$$

26. goods market clearing for good 2:

$$y^T_{2,t}\nu_2 = c_{12,t} + x_{12,t} + (c_{22,t} + x_{22,t})\nu_2$$

is approximated by

$$\hat{y}^T_{2,t} = \frac{c_{12}}{y^T_2\nu_2}\hat{c}_{12,t} + \frac{c_{22}}{y^T_2}\hat{c}_{22,t} + \frac{x_{12}}{y^T_2\nu_2}\hat{x}_{12,t} + \frac{x_{22}}{y^T_2}\hat{x}_{22,t}$$

These equations can be rewritten in terms of the three state variables $\Delta b_{1,t}$, $\hat{k}_{1,t}$, and $\hat{k}_{2,t}$.

C Appendix: Overlapping generations model

This appendix merges the framework with overlapping generations of infinitely lived households by Weil (1989) and Ghironi (2006) with the model presented in the main text. Absent Ricardian equivalence, the steady state level of the net foreign asset position can be shown to be unique. Furthermore, the local dynamics of the net foreign asset position are stationary.

C.1 The model

Each household consumes, supplies labor, and holds financial assets. Households are born on different dates owning no assets, but they own the present discounted value of their labor income. The number of households in country i $N_{i,t}$ grows over time at the exogenous rate n_i, i.e., $N_{i,t+1} = (1+n_i)\, N_{i,t}$. The utility function of each household is assumed to be of the Cobb-Douglas type. Capital is no longer accumulated by households directly, but through capital producers. Households can purchase shares in these and all other firms that produce in their country of residence. To allow the intertemporal elasticity of substitution σ_i to differ from 1 as in Ghironi (2006), aggregation requires perfect foresight.

C.1.1 Individual Households

Consumers have identical preferences over the real consumption index and leisure. At time t_0, the representative consumer in country i born in period $v \in (-\infty, t_0)$ maximizes the intertemporal utility function:

$$U_{t_0}^{v} = \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{\left[\left(c_{i,t}^{v}\right)^{\xi_i} \left(1-l_{i,t}^{v}\right)^{1-\xi_i}\right]^{1-\sigma_i} - 1}{1-\sigma_i} \tag{24}$$

s.t.

$$P_{i,t}^{C} c_{i,t}^{v} \leq P_{i,t} w_{i,t} l_{i,t}^{v} + b_{i,t-1}^{v} - Q_{i,t} b_{i,t}^{v} + \left(Q_{i,t}^{\varpi} + P_{i,t} d_{i,t}\right) \varpi_{i,t-1}^{v} - Q_{i,t}^{\varpi} \varpi_{i,t}^{v}. \tag{25}$$

The final consumption of a representative household of country i born in period v at time t is denoted by $c_{i,t}^{v}$. The production of the final consumption good follows the same process as outlined in the main text. $l_{i,t}^{v}$ is the labor supply of such a household. The bond holdings are denoted by $b^v_{i,t}$. $\varpi^v_{i,t}$ are the household's holdings of the domestic stock, that sells at the price $Q^{\varpi}_{i,t}$ and pays the dividend $P_{i,t}d_{i,t}$. The supply of the stock is normalized to 1.

Household choices for consumption, leisure, and asset holdings must satisfy the following first order conditions:

$$l^v_{i,t} = 1 - \frac{1-\xi_i}{\xi_i}\frac{1}{\Phi_{i,t}w_{i,t}}c^v_{i,t}, \tag{26}$$

$$Q_{i,t} = \beta\left\{\left(\frac{c^v_{i,t+1}}{c^v_{i,t}}\right)^{-\sigma_i}\frac{P^C_{i,t}}{P^C_{i,t+1}}\left(\frac{\Phi_{i,t}w_{i,t}}{\Phi_{i,t+1}w_{i,t+1}}\right)^{(1-\xi_i)(1-\sigma_i)}\right\}, \tag{27}$$

$$1 = \beta\left\{R^{\varpi}_{i,t+1}\left(\frac{c^v_{i,t+1}}{c^v_{i,t}}\right)^{-\sigma_i}\frac{P^C_{i,t}}{P^C_{i,t+1}}\left(\frac{\Phi_{i,t}w_{i,t}}{\Phi_{i,t+1}w_{i,t+1}}\right)^{(1-\xi_i)(1-\sigma_i)}\right\}, \tag{28}$$

where the return of the stock is given by $R^{\varpi}_{i,t+1} = \frac{Q^{\varpi}_{i,t+1}+P_{i,t+1}d_{i,t+1}}{Q^{\varpi}_{i,t}}$.

C.1.2 Firms

While I assume that the production structure of the economy is unchanged relative to the main text, capital is held by capital producers. These firms buy the investment good, augment the existing capital stock, and rent out capital to the producers of traded and non-traded goods. They also pay a dividend to the stockholders. The optimization problem of the capital producers is given by:

$$\max_{k_{i,t},x_{i,t}} \sum_{t=t_0}^{\infty} Q_{i,t_0|t}(1+n_i)^t\left(P_{i,t}r_{i,t}k_{i,t-1} - P^I_{i,t}x_{i,t}\right)$$

$$s.t.$$

$$(1+n_i)k_{i,t} \leq (1-\delta)\,k_{i,t-1} + x_{i,t}. \tag{29}$$

$Q_{i,t_0|t}$ is the pricing kernel used by the capital producers to discount future profits. The capital used in time t production, $k_{i,t-1}$, and investment, $x_{i,t}$, are expressed as per capita averages. The average per capital dividend payment is denoted by $P_{i,t}d_{i,t}$ and satisfies $P_{i,t}d_{i,t} = P_{i,t}r_{i,t}k_{i,t-1} - P^I_{i,t}x_{i,t}$. The optimality condition for the capital producer is therefore given by:

$$\frac{Q_{i,t_0|t+1}}{Q_{i,t_0|t}}\frac{P^c_{i,t+1}}{P^c_{i,t}}\frac{\Phi_{i,t+1}}{\Phi_{i,t}}\frac{\Pi_{i,t}}{\Pi^I_{i,t+1}}\left[\Pi_{i,t+1}r_{i,t+1} + (1-\delta)\right] = 1. \tag{30}$$

C.1.3 Aggregation

If the population at time t is $(1+n_i)^t$, aggregate per capita consumption in country i satisfies

$$c_{i,t} = \frac{1}{(1+n_i)^t}\left[\begin{array}{c} \frac{n_i}{(1+n_i)^{t+1}} c_{i,t}^{-t} + \ldots + \frac{n_i}{(1+n_i)^2} c_{i,t}^{-1} + \frac{n_i}{1+n_i} c_{i,t}^{0} \\ + n c_{i,t}^{1} + n_i (1+n_i) c_{i,t}^{2} + \ldots + n_i (1+n_i)^{t-1} c_{i,t}^{t} \end{array}\right].$$

Aggregate per capita labor supply equations are obtained by aggregating labor-leisure tradeoff equations across generations and dividing by the total population at each point in time. The aggregate per capita labor-leisure tradeoff satisfies

$$1 - l_{i,t} = \frac{1-\zeta_i}{\zeta_i}\frac{1}{\Phi_{i,t} w_{i,t}} c_{i,t}. \tag{31}$$

From (27) and (31) the price of the bond can be shown to satisfy

$$Q_{i,t} = \beta\left(\frac{\Phi_{i,t} w_{i,t}}{\Phi_{i,t+1} w_{i,t+1}}\right)^{(1-\xi_i)(1-\sigma_i)}\left(\frac{c_{i,t+1} - \frac{n_i}{1+n_i} c_{i,t+1}^{t+1}}{\frac{1}{1+n_i} c_{i,t}}\right)^{-\sigma_i}\frac{P_{i,t}^C}{P_{i,t+1}^C} \tag{32}$$

under perfect foresight.

To obtain the aggregate per capita budget constraint, note that the stock is in fixed supply. Thus,

$$c_{i,t} = \Phi_{i,t} w_{i,t} l_{i,t} + \Phi_{i,t} r_{i,t} k_{i,t-1} - \Pi_{i,t} x_{i,t} + \frac{1}{P_{i,t}^C}\left\{b_{i,t-1} - (1+n_i) Q_{i,t} b_{i,t}\right\}. \tag{33}$$

Remains to determine the consumption of the newly born generation in period t. Defining total asset holdings of generation v in period t by

$$a_{i,t}^v = \frac{1}{P_{i,t}^C} b_{i,t}^v + \frac{1}{Q_{i,t}}\frac{Q_{i,t}^{\varpi}}{P_{i,t}^C}\varpi_{i,t}^v, \tag{34}$$

the budget constraint of a generation v household is written as

$$Q_{i,t} a_{i,t}^v \leq a_{i,t-1}^v + \Phi_{i,t} w_{i,t} l_{i,t}^v - c_{i,t}^v. \tag{35}$$

Iterating forward on equation (35) and invoking the appropriate transversality condition, the budget constraint can be shown to satisfy

$$0 = a_{i,t-1}^v + \Phi_{i,t} w_{i,t}\Theta_{i,t}^2 - \left\{c_{i,t}^v + \Phi_{i,t} w_{i,t}\left(1 - l_{i,t}^v\right)\right\}\Theta_{i,t}^1 \tag{36}$$

with

$$\Theta^1_{i,t} = 1 + Q_{i,t}\left(\frac{Q_{i,t}}{\beta}\frac{P^C_{i,t+1}}{P^C_{i,t}}\right)^{-\frac{1}{\sigma_i}}\left(\frac{\Phi_{i,t}w_{i,t}}{\Phi_{i,t+1}w_{i,t+1}}\right)^{\frac{(1-\xi_i)(1-\sigma_i)}{\sigma_i}}\Theta^1_{i,t+1}, \tag{37}$$

$$\Theta^2_{i,t} = 1 + Q_{i,t}\frac{\Phi_{i,t+1}w_{i,t+1}}{\Phi_{i,t}w_{i,t}}\Theta^2_{i,t+1}, \tag{38}$$

using the fact that

$$\begin{aligned} &\left(\frac{Q_{i,t+s-1}...Q_{i,t+1}Q_{i,t}}{\beta^s}\frac{P^C_{i,t+s}}{P^C_{i,t}}\right)^{-\frac{1}{\sigma_i}}\left(\frac{\Phi_{i,t}w_{i,t}}{\Phi_{i,t+s}w_{i,t+s}}\right)^{\frac{(1-\xi_i)(1-\sigma_i)}{\sigma_i}}\left\{c^v_{i,t} + \Phi_{i,t}w_{i,t}\left(1-l^v_{i,t}\right)\right\} \\ = \quad & c^v_{i,t+s} + \Phi_{i,t+s}w_{i,t+s}\left(1-l^v_{i,t+s}\right). \end{aligned} \tag{39}$$

As newly born agents do not hold financial assets, their consumption is given by

$$c^t_{i,t} = \xi_i\Phi_{i,t}w_{i,t}\frac{\Theta^2_{i,t}}{\Theta^1_{i,t}}. \tag{40}$$

Note, in general the consumption of newly born households differs from the aggregate per capita consumption. Hence, the steady states of the model with overlapping generations differ from those obtained in the representative agents model.

D Appendix: nominal and real rigidities

This appendix lays out the model with nominal and real rigidities. Following the influential work of Christiano, Eichenbaum, and Evans (1999) and Smets and Wouters (2003), the model features sticky wages and prices, consumption habits, and investment adjustment costs. Wage and price contracts are modeled as in Calvo (1983) and Yun (1996). Unless noted otherwise, the modeling features introduced in the main text remain unchanged. As in the main text, variables are expressed in per capita terms.

D.1 Additional modeling features

Wages A continuum of monopolistically competitive households (indexed on the unit interval) supplies differentiated labor services to the intermediate goods-producing sector. A representative labor aggregator combines the households' labor hours in the same proportions as firms would choose. This labor index l_{t+j} has the Dixit-Stiglitz form:

$$l_{t+j} = \left[\int_0^1 l_{t+j}\left(h\right)^{\frac{1}{1+\theta_i^w}} dh \right]^{1+\theta_i^w}, \tag{41}$$

where $\theta_i^w > 0$ and $l_t(h)$ is hours worked by a typical member of household h. The aggregator minimizes the cost of producing a given amount of the aggregate labor index, taking each household's wage rate $W_{t+j}\left(h\right)$ as given. One unit of the labor index sells at the unit cost W_{t+j}:

$$W_{t+j} = \left[\int_0^1 W_{t+j}\left(h\right)^{\frac{-1}{\theta_i^w}} dh \right]^{-\theta_i^w}. \tag{42}$$

W_{t+j} is referred to as the aggregate wage index. The aggregator's demand for the labor services of household h satisfies:

$$l_{i,t+j}\left(h\right) = \left[\frac{W_{i,t+j}\left(h\right)}{W_{i,t+j}} \right]^{-\frac{1+\theta_i^w}{\theta_i^w}} l_{i,t+j}. \tag{43}$$

The utility functional of a typical member of household h is:

$$\begin{aligned} &\widetilde{\mathbb{E}}_t \sum_{j=0}^{\infty} \beta_i^j \left\{ \frac{1}{1-\sigma_i} \left(c_{i,t+j}\left(h\right) - \kappa_i c_{i,t+j-1} - \nu_{i,t+j} \right)^{1-\sigma_i} \right. \\ &\left. -\frac{\chi_{0i}}{1-\chi_i} \left(l_{i,t+j}\left(h\right) \right)^{1-\chi_i} + V\left(\frac{mb_{i,t+j+1}\left(h\right)}{P_{i,t+j}^C} \right) \right\}, \end{aligned} \tag{44}$$

where the discount factor β_i satisfies $0 < \beta_i < 1$. As in Smets and Wouters (2003), I allow for the possibility of external habits. At date $t+j$, a member of household h cares about consumption relative to lagged per capita consumption $c_{i,t+j-1}$. $\nu_{i,t+j}$ is a preference shock that follows an AR(1) process. Furthermore, the period utility function depends on labor $l_{i,t+j}\left(h\right)$ and the end-of-period real money balances, $\frac{mb_{i,t+j+1}(h)}{P_{i,t+j}^C}$. The budget constraint of a member of household h is given by:

$$
\begin{aligned}
& P^C_{i,t+j} c_{i,t+j}(h) + P^I_{i,t+j} x_{i,t+j}(h) + P^I_{i,t+j} \frac{1}{2} \phi^I_i \frac{\left(x_{i,t+j} - x_{i,t-1+j}(h)\right)^2}{x_{i,t-1+j}(h)} \\
& + \int_S \zeta_{t+j,t+1+j} b^D_{i,t+j,t+1+j}(h) - b^D_{i,t-1+j,t+j}(h) + P^G_{i,t+j} B^G_{i,t+j} - B^G_{i,t-1+j} \\
& + Q_{i,t+j} B_{i,t+j}(h) - B_{i,t-1+j}(h) + P^C_{i,t+j} \Gamma\left(\frac{B_{i,t+j}}{P^C_{i,t+j}}\right) \\
\leq \; & (1 - \tau^w_i) W_{i,t+j}(h) l_{i,t+j}(h) + (1 - \tau^r_i) R_{i,t+j} k_{i,t-1+j}(h) \\
& + Pr_{i,t+j} + Tr_{i,t+j}. \qquad (45)
\end{aligned}
$$

The left hand side of the budget constraint summarizes the expenditures for consumption, investment (plus investment adjustment costs), expenditure and income from domestically-issued and traded bonds, government bonds, and the internationally-traded bond (plus portfolio cost). The right hand side provides information about the agent's labor and capital income, profits, and transfers.

Household wages are determined by Calvo-style staggered contracts subject to static wage indexation. In particular, with probability $1 - \xi^w_i$, each household is allowed to reoptimize its wage contract. If a household is not allowed to reoptimize its wage rate, it sets its wage according to $W_{i,t+j}(h) = \bar{\omega}_{i,t-1+j} W_{i,t-1+j}(h)$ with $\bar{\omega}_{i,t+j} = \omega^{\gamma^w_i}_{i,t+j} \pi^{*1-\gamma^w_i}$. $\omega_{i,t+j}$ is the actual wage inflation rate and π^* is the steady state inflation rate. γ^w_i is referred to as the wage indexation parameter. A household who is resetting its wage in period t chooses $W^*_{i,t}(h)$ such that:

$$
\begin{aligned}
0 \; = \; & -\mathbb{E}_t \sum_{j=0}^{\infty} (\beta_i \xi^w_i)^j \left[\chi_{0i} (l_{i,t+j}(h))^{-\chi_i} \left(-\frac{1+\theta^w_i}{\theta^w_i}\right) \left[\frac{V^W_{i,t,j} W^*_{i,t}(h)}{W_{i,t+j}}\right]^{-\frac{1+\theta^w_i}{\theta^w_i}} \frac{l_{i,t+j}}{W^*_{i,t}(h)} \right] \\
& + \mathbb{E}_t \sum_{j=0}^{\infty} (\beta_i \xi^w_i)^j \left[(1 - \tau^w_i) \lambda_{i,t+j}(h) V^W_{i,t,j} W^*_{i,t}(h) \left(-\frac{1+\theta^w_i}{\theta^w_i}\right) \left[\frac{V^W_{i,t,j} W^*_{i,t}(h)}{W_{i,t+j}}\right]^{-\frac{1+\theta^w_i}{\theta^w_i}} \frac{l_{i,t+j}}{W^*_{i,t}(h)} \right] \\
& + \mathbb{E}_t \sum_{j=0}^{\infty} (\beta_i \xi^w_i)^j \lambda_{i,t+j}(h) (1 - \tau^w_i) V^W_{i,t,j} l_{i,t+j}(h),
\end{aligned}
$$

where $V^W_{i,t,0} = 1$ and $V^W_{i,t,j} = \prod_{k=1}^{j} \bar{\omega}_{i,t+k-1}$ for $j > 0$. Since households have access to complete insurance markets domestically, the marginal utility of wealth $\lambda_{i,t+j}(h)$ is constant across households. Thus, I drop the dependence of this variable on h. Furthermore, the assumption of complete markets implies that all households that reoptimize in period t choose the same wage rate $W_{i,t}^{*}(h) = W_{i,t}^{*}$ Hence:

$$\frac{H_{i,t}^W}{G_{i,t}^W} = \left(\frac{W_{i,t}^*}{P_{ii,t}}\right)^{1-\frac{1+\theta_i^w}{\theta_i^w}\chi_i} \tag{46}$$

$$\begin{aligned}
H_{i,t}^W &= \mathbb{E}_t \sum_{j=0}^{\infty} \left(\beta_i \xi_i^w\right)^j \chi_{0i} \left(\left[V_{i,t,j}^W \frac{P_{ii,t+j}}{W_{i,t+j}} \frac{P_{ii,t}}{P_{ii,t+j}} \right]^{-\frac{1+\theta_i^w}{\theta_i^w}} l_{i,t+j} \right)^{1-\chi_i} \\
&= \chi_{0i} \left(\left[\frac{P_{ii,t}}{W_{i,t}} \right]^{-\frac{1+\theta_i^w}{\theta_i^w}} l_{i,t} \right)^{1-\chi_i} + \left(\beta_i \xi_i^w\right)^1 \mathbb{E}_t \left(\frac{\bar{\omega}_{i,t}}{\pi_{ii,t+1}} \right)^{-\frac{1+\theta_i^w}{\theta_i^w}(1-\chi_i)} H_{i,t+1}^W \\
G_{i,t}^W &= \mathbb{E}_t \sum_{j=0}^{\infty} \left(\beta_i \xi_i^w\right)^j \frac{1-\tau_i^w}{1+\theta_i^w} \lambda_{i,t+j} P_{i,t+j}^C \Phi_{i,t+j} \left(\frac{P_{ii,t}}{P_{ii,t+j}} V_{i,t,j}^W \right) \left[V_{i,t,j}^W \frac{P_{ii,t+j}}{W_{i,t+j}} \frac{P_{ii,t}}{P_{ii,t+j}} \right]^{-\frac{1+\theta_i^w}{\theta_i^w}} l_{i,t+j} \\
&= \frac{1-\tau_i^w}{1+\theta_i^w} \lambda_{i,t} P_{i,t}^C \Phi_{i,t} \left[\frac{P_{ii,t}}{W_{i,t}} \right]^{-\frac{1+\theta_i^w}{\theta_i^w}} l_{i,t} + \beta_i \xi_i^w \mathbb{E}_t \left(\frac{\bar{\omega}_{i,t}}{\pi_{ii,t+1}} \right)^{\frac{-1}{\theta_i^w}} G_{i,t+1}^W.
\end{aligned}$$

The real wage expressed in terms of country i's traded good evolves according to

$$\frac{W_{i,t+j}}{P_{ii,t+j}} = \left[\left(1-\xi_i^w\right) \left(\frac{W_{i,t+j}^*}{P_{ii,t+j}} \right)^{\frac{-1}{\theta_i^w}} + \xi_i^w \left(\frac{\bar{\omega}_{i,t+j-1}}{\pi_{ii,t+j}} \frac{W_{i,t+j-1}}{P_{ii,t+j-1}} \right)^{\frac{-1}{\theta_i^w}} \right]^{-\theta_i^w} \tag{47}$$

with $\pi_{ii,t+j} = \frac{P_{ii,t+j}}{P_{ii,t+j-1}}$.

Prices Both for traded and non-traded goods, there is a continuum of differentiated intermediate goods in country i indexed by $h \in [0, 1]$, each produced by a single monopolistically competitive firm. Deviating from the main text, I assume that individual varieties are traded internationally rather than the final aggregate. Goods with index $h \in [0, \bar{h}_i]$ are traded. Consumption and investment varieties are interchangeable. The use of the index h in the description of prices is unrelated to its use in the description of wages.

Let $y_i^T(h)$ be the overall quantity produced by firm h of the traded goods producers using a CES technology. Firms operate in perfectly competitive factor markets for aggregate capital and the labor aggregate and thus operate with the same capital to labor ratio. Denoting individual labor and capital demands by $l^T_{i,t+j}(h)$ and $k^T_{i,t+j}(h)$, respectively, and aggregate labor and capital demand by $l^T_{i,t+j} = \int_0^1 l^T_{i,t+j}(h)\,dh$ and $k^T_{i,t+j} = \int_0^1 k^T_{i,t+j}(h)\,dh$, I define aggregate output as:

$$y^T_{i,t+j} = \int_0^1 y^T_{i,t+j}(h)\,dh = \left[\left(\omega^T_{li}\right)^{1-\kappa^T}\left(\frac{\omega^T_{ki}}{\omega^T_{li}}\right)^{\kappa^T}\left(\frac{w_{i,t+j}}{A^T_{i,t+j}r_{i,t+j}}\right)^{\frac{\kappa^T}{\kappa^T-1}} + \left(\omega^T_{ki}\right)^{1-\kappa^T}\right]^{\frac{1}{\kappa^T}} k^T_{i,t+j}$$

if $\kappa^T_i < 1$, and

$$y^T_{i,t+j} = \int_0^1 y^T_{i,t+j}(h)\,dh = \frac{1}{1-\omega^T_{li}}\left(A^T_{i,t+j}\frac{r_{i,t+j}}{w_{i,t+j}}\right)^{\omega^T_{li}} k^T_{i,t+j}$$

if $\kappa^T_i = 0$.

Home and foreign prices of the intermediate goods are determined by Calvo-style staggered contracts. Each period, a firm of country 1's traded goods sector faces a constant probability $1-\xi^{pT}_{11}$ of being able to reoptimize its price at home and $1-\xi^{pT}_{21}$ probability of being able to reoptimize its price abroad. These probabilities are assumed to be independent across firms, time, and countries. For country 2 the respective probabilities are $1-\xi^{pT}_{12}$ and $1-\xi^{pT}_{22}$. Similar to the behavior of wages, firms that are not allowed to reoptimize their price in the current period, adjust their price by a geometric average of the actual inflation rate of the previous period (with weight γ^{pT}_{im}, $i = 1,2$ and $m = 1,2$) and the steady state inflation rate π^* (with weight $1-\gamma^{pT}_{im}$, $i = 1,2$ and $m = 1,2$). I denote this average by $\bar{\pi}^{pT}_{im} = \pi^{pT\gamma^{pT}_{im}}_{im}\pi^{*1-\gamma^{pT}_{im}}$.

I define the following aggregates of consumer and investment demand for the varieties produced in country 1:

$$c_{11,t+j} = \left[\int_0^1 c_{11,t+j}(h)^{\frac{1}{1+\theta^{pT}_1}}dh\right]^{1+\theta^{pT}_1} \text{ and } c_{21,t+j} = \bar{h}_1^{-\theta^{pT}_1}\left[\int_0^{\bar{h}_1} c_{21,t+j}(h)^{\frac{1}{1+\theta^{pT}_1}}dh\right]^{1+\theta^{pT}_1},$$

$$x_{11,t+j} = \left[\int_0^1 x_{11,t+j}(h)^{\frac{1}{1+\theta^{pT}_1}}dh\right]^{1+\theta^{pT}_1} \text{ and } x_{21,t+j} = \bar{h}_1^{-\theta^{pT}_1}\left[\int_0^{\bar{h}_1} x_{21,t+j}(h)^{\frac{1}{1+\theta^{pT}_1}}dh\right]^{1+\theta^{pT}_1},$$

and the associated demand equations:

$$c_{11,t+j}(h) = \left[\frac{P_{11,t+j}(h)}{P_{11,t+j}}\right]^{\frac{-\left(1+\theta^{pT}_1\right)}{\theta^{pT}_1}} c_{11,t+j} \text{ and } c_{21,t+j}(h) = \frac{1}{\bar{h}_1}\left(\frac{P_{21,t+j}(h)}{P_{21,t+j}}\right)^{-\frac{1+\theta^{pT}_1}{\theta^{pT}_1}} c_{21,t+j},$$

$$x_{11,t+j}(h) = \left[\frac{P_{11,t+j}(h)}{P_{11,t+j}}\right]^{\frac{-\left(1+\theta^{pT}_1\right)}{\theta^{pT}_1}} x_{11,t+j} \text{ and } x_{21,t+j}(h) = \frac{1}{\bar{h}_1}\left(\frac{P_{21,t+j}(h)}{P_{21,t+j}}\right)^{-\frac{1+\theta^{pT}_1}{\theta^{pT}_1}} x_{21,t+j}.$$

The aggregator $P_{11,t+j} = \left[\int_0^1 \left[P_{11,t+j}(h)\right]^{-\frac{1}{\theta_1^{pT}}} dh\right]^{-\theta_1^{pT}}$ for domestically produced varieties implies:

$$1 = \left(1-\xi_{11}^{pT}\right)\left[\frac{P^*_{11,t+j}}{P_{11,t+j}}\right]^{-\frac{1}{\theta_1^{pT}}} + \xi_{11}^{pT}\left[\frac{\bar{\pi}_{11,t+j-1}}{\pi_{11,t+j}}\right]^{-\frac{1}{\theta_1^{pT}}},$$

where $P^*_{11,t+j}$ is the price set by those firms that reoptimize in period $t+j$ and $\pi_{11,t+j} = \frac{P_{11,t+j}}{P_{11,t+j-1}}$. Reoptimizing firms choose to set the same price since marginal costs are identical across firms.

With respect to the export price index, $P_{21,t+j}$ denotes the price index in the currency of country 2 (local currency) and $\tilde{P}_{21,t+j}$ denotes the price index in the currency of country 1 (exporter currency):

$$\begin{aligned}
P_{21,t+j} &= \left[\frac{1}{\bar{h}_1}\int_0^{\bar{h}_1}\left(P_{21,t+j}(h)\right)^{-\frac{1}{\theta_1^{pT}}} dh\right]^{-\theta_1^{pT}}, \\
1 &= \left[\left(1-\xi_{21}^{pT}\right)\left[\frac{P^*_{21,t+j}}{P_{21,t+j}}\right]^{-\frac{1}{\theta_1^{pT}}} + \xi_{21}^{pT}\left(\frac{\bar{\pi}_{21,t+j-1}}{\pi_{21,t+j}}\right)^{-\frac{1}{\theta_1^{pT}}}\right]^{-\theta_1^{pT}}, \\
\tilde{P}_{21,t+j} &= \left[\frac{1}{\bar{h}_1}\int_0^{\bar{h}_1}\left(\tilde{P}_{21,t+j}(h)\right)^{-\frac{1}{\theta_1^{pT}}} dh\right]^{-\theta_1^{pT}}, \\
1 &= \left[\left(1-\xi_{21}^{pT}\right)\left[\frac{\tilde{P}^*_{21,t+j}}{\tilde{P}_{21,t+j}}\right]^{-\frac{1}{\theta_1^{pT}}} + \xi_{21}^{pT}\left(\frac{\bar{\tilde{\pi}}_{21,t+j-1}}{\tilde{\pi}_{21,t+j}}\right)^{-\frac{1}{\theta_1^{pT}}}\right]^{-\theta_1^{pT}},
\end{aligned}$$

where $\pi_{21,t+j} = \frac{P_{21,t+j}}{P_{21,t+j-1}}$ and $\tilde{\pi}_{21,t+j} = \frac{\tilde{P}_{21,t+j}}{\tilde{P}_{21,t+j-1}}$. In addition, I define the price dispersion functions:

$$\begin{aligned}
\Delta_{11,t+j} &= \int_0^1 \left[\frac{P_{11,t+j}(h)}{P_{11,t+j}}\right]^{\frac{-\left(1+\theta_1^{pT}\right)}{\theta_1^{pT}}} dh, \\
\Delta_{11,t+j} &= \left(1-\xi_{11}^{pT}\right)\left[\frac{P^*_{11,t+j}}{P_{11,t+j}}\right]^{\frac{-\left(1+\theta_1^{pT}\right)}{\theta_1^{pT}}} + \xi_{11}^{pT}\left[\frac{\bar{\pi}_{11,t+j-1}}{\pi_{11,t+j}}\right]^{\frac{-\left(1+\theta_1^{pT}\right)}{\theta_1^{pT}}} \Delta_{11,t+j-1}, \\
\Delta_{21,t+j} &= \frac{1}{\bar{h}_1}\int_0^{\bar{h}_1}\left(\frac{P_{21,t+j}(h)}{P_{21,t+j}}\right)^{-\frac{1+\theta_1^{pT}}{\theta_1^{pT}}} dh, \\
\Delta_{21,t+j} &= \left(1-\xi_{21}^{pT}\right)\left[\frac{P^*_{21,t+j}}{P_{21,t+j}}\right]^{-\frac{1+\theta_1^{pT}}{\theta_1^{pT}}} + \xi_{21}^{pT}\left[\frac{\bar{\pi}_{21,t+j-1}}{\pi_{21,t+j}}\right]^{-\frac{1+\theta_1^{pT}}{\theta_1^{pT}}} \Delta_{21,t+j-1},
\end{aligned}$$

and equivalently for $\tilde{\Delta}_{21,t+j}$. As the market clearing condition for each variety h satisfies:

$$y_{1,t+j}^{T}(h) = \left[c_{11,t+j}(h) + x_{11,t+j}(h)\right] + \left[c_{21,t+j}(h) + x_{21,t+j}(h)\right] \upsilon_2,$$

in the aggregate one obtains under the assumption of local currency pricing:

$$\begin{aligned} y_{1,t+j}^{T} &= \left(\int_0^1 \left[\frac{P_{11,t+j}(h)}{P_{11,t+j}}\right]^{\frac{-\left(1+\theta_1^{pT}\right)}{\theta_1^{pT}}} dh\right) \left[c_{11,t+j} + x_{11,t+j}\right] \\ &\quad + \left(\frac{1}{\bar{h}_1}\int_0^{\bar{h}_1} \left(\frac{P_{21,t+j}(h)}{P_{21,t+j}}\right)^{-\frac{1+\theta_1^{pT}}{\theta_1^{pT}}} dh\right) \left[c_{21,t+j} + x_{21,t+j}\right] \\ &= \Delta_{11,t+j}\left[c_{11,t+j} + x_{11,t+j}\right] + \Delta_{21,t+j}\left[c_{21,t+j} + x_{21,t+j}\right]. \end{aligned}$$

The parameter υ_2 adjusts for the differences in country size.

As before the consumption of traded goods requires distribution services in the form of the non-traded goods. Market clearing for the non-traded good is derived from:

$$\begin{aligned} y_{i,t+j}^{N}(h) &= c_{i,t+j}^{N}(h) + \eta\left(c_{i1,t+j}(h) + c_{i2,t+j}(h)\right), \\ y_{i,t+j}^{N} &= \int y_{i,t+j}^{N}(h)\, dh = \Delta_{i,t}^{N} c_{i,t}^{N} + \eta\left(c_{i1,t+j} + c_{i2,t+j}\right), \end{aligned}$$

where

$$\Delta_{i,t}^{N} = \int \left[\frac{P_{i,t}(h)}{P_{i,t}}\right]^{\frac{-\left(1+\theta_n^{pN}\right)}{\theta_n^{pN}}} dh$$

which follows a similar law of motion as the dispersion measure of traded goods prices.

Remains to define the optimization problem of firms. A traded goods producer firm located in country 1 that sells domestically chooses its price $P_{11,t}(h)$ to maximize:

$$E_t \sum_{j=0}^{\infty} \xi_{11}^{pT} \psi_{1,t,t+j} \left[V_{11,t,j}^{pT}\left(1+\tau_{11}\right) P_{11,t}(h)\, y_{11,t+j}^{T}(h) - MC_{1,t+j} y_{11,t+j}^{T}(h)\right], \tag{48}$$

where

$$
\begin{aligned}
y^{T}_{11,t+j}(h) &= c_{11,t+j}(h) + x_{11,t+j}(h), && (49)\\
V^{pT}_{11,t,j} &= \prod_{k=1}^{j} \bar{\pi}_{11,t+k-1}, && (50)\\
y^{T}_{11,t+j}(h) &= \left[\frac{V^{pT}_{11,t,j} P_{11,t}(h)}{P_{11,t+j}}\right]^{\frac{-\left(1+\theta^{pT}_{1}\right)}{\theta^{pT}_{1}}} y^{T}_{11,t+j}. && (51)
\end{aligned}
$$

τ_{11} is a price subsidy. $\psi_{1,t,t+j}$ is the stochastic discount factor that relates to the marginal utility of wealth as defined below. Using $\frac{\partial y^{T}_{11,t+j}(h)}{\partial P_{11,t}(h)} = \frac{-\left(1+\theta^{pT}_{1}\right)}{\theta^{pT}_{1}} \frac{y^{T}_{11,t+j}(h)}{P_{11,t}(h)}$, the first order condition implies:

$$
E_t \sum_{j=0}^{\infty} \left(\xi^{pT}_{11}\right)^{j} \psi_{1,t,t+j} \left[V^{pT}_{11,t,j} \frac{1+\tau_{11}}{1+\theta^{pT}_{1}} P^{*}_{11,t} - MC_{1,t+j}\right] \frac{y^{T}_{11,t+j}(h)}{P^{*}_{11,t}} = 0, \tag{52}
$$

which can also be written as:

$$
\begin{aligned}
\frac{P^{*}_{11,t}}{P_{11,t}} &= \frac{H^{pT}_{11,t}}{G^{pT}_{11,t}}, && (53)\\
H^{pT}_{11,t} &= E_t \sum_{j=0}^{\infty} \left(\xi^{pT}_{11}\right)^{j} \psi_{1,t,t+j} \frac{P_{11,t+j}}{P_{11,t}} \frac{MC_{1,t+j}}{P_{11,t+j}} \left(\frac{V^{pT}_{11,t,j} P_{11,t}}{P_{11,t+j}}\right)^{\frac{-\left(1+\theta^{pT}_{1}\right)}{\theta^{pT}_{1}}} y^{T}_{11,t+j} \\
&= \frac{MC_{1,t}}{P_{11,t}} y^{T}_{11,t} + \xi^{pT}_{11} E_t \left\{ \psi_{1,t,t+1} \pi_{11,t+1} \left(\frac{\bar{\pi}_{11,t}}{\pi_{11,t+1}}\right)^{\frac{-\left(1+\theta^{pT}_{1}\right)}{\theta^{pT}_{1}}} H^{pT}_{11,t+1} \right\}, && (54)\\
G^{pT}_{11,t} &= E_t \sum_{j=0}^{\infty} \left(\xi^{pT}_{11}\right)^{j} \psi_{1,t,t+j} V^{pT}_{11,t,j} \frac{1+\tau_{11}}{1+\theta^{pT}_{1}} \left(\frac{V^{pT}_{11,t,j} P_{11,t}}{P_{11,t+j}}\right)^{\frac{-\left(1+\theta^{pT}_{1}\right)}{\theta^{pT}_{1}}} y^{T}_{11,t+j} \\
&= \frac{1+\tau_{11}}{1+\theta^{pT}_{1}} y^{T}_{11,t} + \xi^{pT}_{11} E_t \left\{ \psi_{1,t,t+1} \pi_{11,t+1} \left(\frac{\bar{\pi}_{11,t}}{\pi_{11,t+1}}\right)^{1-\frac{\left(1+\theta^{pT}_{1}\right)}{\theta^{pT}_{1}}} G^{pT}_{11,t+1} \right\}, && (55)\\
\psi_{1,t,t+1} &= \beta \frac{\lambda_{t+1} P^{C}_{t+1}}{\lambda_t P^{C}_{t}} \frac{P^{C}_{t}}{P^{C}_{t+1}}. && (56)
\end{aligned}
$$

With respect to export pricing, I distinguish possibilities: producer currency pricing and local currency pricing. If country 1's exporters engage in producer currency pricing:

$$E_t \sum_{j=0}^{\infty} \xi_{21}^{pT} \psi_{1,t,t+j} \left[(1+\tau_{21}) \tilde{V}_{21,t,j}^{pT} \tilde{P}_{21,t}(h) - MC_{1,t+j} \right] y_{21,t+j}^{T}(h), \tag{57}$$

where

$$y_{21,t+j}^{T}(h) = c_{21,t+j}(h) + x_{21,t+j}(h), \tag{58}$$

$$\tilde{V}_{21,t,j}^{pT} = \prod_{k=1}^{j} \bar{\tilde{\pi}}_{21,t+k-1}, \tag{59}$$

$$y_{21,t+j}^{T}(h) = \frac{1}{\bar{h}_1} \left(\frac{\tilde{V}_{21,t,j}^{pT} \tilde{P}_{21,t}(h)}{e_{1,t+j} P_{21,t+j}} \right)^{\frac{-(1+\theta_1^{pT})}{\theta_1^{pT}}} y_{21,t+j}^{T}. \tag{60}$$

Note, that I have defined $\tilde{\pi}_{21,t} = \frac{\tilde{P}_{21,t}}{\tilde{P}_{21,t-1}}$ and $\frac{\tilde{P}_{21,t}(h)}{e_{1,t+j}} = P_{21,t}(h)$. Using $\frac{\partial y_{21,t+j}^{T}(h)}{\partial \tilde{P}_{21,t}(h)} = -\frac{1+\theta_1^{pT}}{\theta_1^{pT}} \frac{y_{21,t+j}^{T}(h)}{\tilde{P}_{21,t}(h)}$, the first order condition implies:

$$E_t \sum_{j=0}^{\infty} \left(\xi_{21}^{pT} \right)^j \psi_{1,t,t+j} \left[\frac{1+\tau_{21}}{1+\theta_1^{pT}} \tilde{V}_{21,t,j}^{pT} \tilde{P}_{21,t}^{*} - MC_{1,t+j} \right] \frac{y_{21,t+j}^{T}(h)}{\tilde{P}_{21,t}^{*}} = 0,$$

or

$$\begin{aligned} \frac{\tilde{P}_{21,t}^{*}}{\tilde{P}_{21,t}} &= \frac{\tilde{H}_{21,t}^{pT}}{\tilde{G}_{21,t}^{pT}} \\ \tilde{H}_{21,t}^{pT} &= \frac{MC_{1,t}}{P_{11,t}} \frac{P_{11,t}}{e_{1,t} P_{21,t}} y_{21,t}^{T} + \xi_{21}^{pT} E_t \left\{ \psi_{1,t,t+1} \tilde{\pi}_{21,t+1} \left(\frac{\bar{\tilde{\pi}}_{21,t}}{\tilde{\pi}_{21,t+1}} \right)^{\frac{-(1+\theta_1^{pT})}{\theta_1^{pT}}} \tilde{H}_{21,t+1}^{pT} \right\}, \\ \tilde{G}_{21,t}^{pT} &= \frac{1+\tau_{21}}{1+\theta_1^{pT}} y_{21,t}^{T} + \xi_{21}^{pT} E_t \left\{ \psi_{1,t,t+1} \tilde{\pi}_{21,t+1} \left(\frac{\bar{\tilde{\pi}}_{21,t}}{\tilde{\pi}_{21,t+1}} \right)^{1-\frac{(1+\theta_1^{pT})}{\theta_1^{pT}}} \tilde{G}_{21,t+1}^{pT} \right\}. \end{aligned}$$

Under local currency pricing, a firm maximizes:

$$E_t \sum_{j=0}^{\infty} \left(\xi_{21}^{pT} \right)^j \psi_{1,t,t+j} \left[(1+\tau_{21}) V_{21,t,j}^{pT} P_{21,t}(h) e_{1,t+j} - MC_{1,t+j} \right] y_{21,t+j}(h), \tag{61}$$

where

$$V_{21,t,j}^{pT} = \prod_{k=1}^{j} \bar{\pi}_{21,t+k-1}, \tag{62}$$

$$V_{1,t,j}^{e} = \prod_{k=1}^{j} \pi_{1,t+k-1}^{e}, \tag{63}$$

$$y_{21,t+j}(h) = \frac{1}{\bar{h}_1} \left(\frac{V_{21,t,j}^{pT} P_{21,t}(h)}{P_{21,t+j}} \right)^{-\frac{1+\theta_1^{pT}}{\theta_1^{pT}}} y_{21,t+j}. \tag{64}$$

Note, that I have used the definitions $\pi_{21,t} = \frac{P_{21,t}}{P_{21,t-1}}$, $\frac{\tilde{P}_{21,t}(h)}{e_{1,t+j}} = P_{21,t}(h)$, $\pi_{1,t}^{e} = \frac{e_{1,t}}{e_{1,t-1}}$. Using $\frac{\partial y_{21,t+j}(h)}{\partial P_{21,t}(h)} = -\frac{1+\theta_1^{pT}}{\theta_1^{pT}} \frac{y_{21,t+j}(h)}{P_{21,t}(h)}$, the first order condition implies:

$$E_t \sum_{j=0}^{\infty} \left(\xi_{21}^{pT} \right)^j \psi_{1,t,t+j} \left[\frac{1+\tau_{21}}{1+\theta_1^{pT}} V_{21,t,j}^{pT} P_{21,t}^{*} e_{1,t+j} - MC_{1,t+j} \right] \frac{y_{21,t+j}(h)}{P_{21,t}^{*}} = 0, \tag{65}$$

or

$$\frac{P_{21,t}^{*}}{P_{21,t}} = \frac{H_{21,t}^{pT}}{G_{21,t}^{pT}}$$

$$H_{21,t}^{pT} = \frac{MC_{1,t}}{P_{11,t}} \frac{P_{11,t}}{e_{1,t} P_{21,t}} y_{21,t}^{T} + \xi_{21}^{pT} E_t \left\{ \psi_{1,t,t+1} \pi_{21,t+1} \left(\frac{\bar{\pi}_{21,t}}{\pi_{21,t+1}} \right)^{\frac{-(1+\theta_1^{pT})}{\theta_1^{pT}}} H_{21,t+1}^{pT} \right\},$$

$$G_{21,t}^{pT} = \frac{1+\tau_{21}}{1+\theta_1^{pT}} y_{21,t}^{T} + \xi_{21}^{pT} E_t \left\{ \psi_{1,t,t+1} \pi_{21,t+1} \left(\frac{\bar{\pi}_{21,t}}{\pi_{21,t+1}} \right)^{1-\frac{(1+\theta_1^{pT})}{\theta_1^{pT}}} G_{21,t+1}^{pT} \right\}.$$

Similar expressions apply for country 2 and the non-traded goods sector.

D.2 Model equations

D.2.1 Definitions

The following definitions are adopted.

1. Real relative prices:

$$
\begin{aligned}
q_{1,t+j} &= \frac{P_{12,t+j}}{P_{11,t+j}}, \\
q_{2,t+j} &= \frac{P_{22,t+j}}{P_{21,t+j}}, \\
q_{t+j} &= \frac{e_{1,t+j}P_{22,t+j}}{P_{11,t+j}}, \\
rer_{t+j} &= \frac{e_{1,t+j}P^C_{2,t+j}}{P^C_{1,t+j}}, \\
q^{N*}_{i,t+j} &= \frac{P^{N*}_{i,t+j}}{P_{ii,t+j}}, \\
\Phi_{i,t+j} &= \frac{P_{ii,t}}{P^C_{i,t}}, \\
\Pi_{i,t+j} &= \frac{P_{ii,t}}{P^I_{i,t}}.
\end{aligned}
$$

2. Optimized relative prices:

$$
\begin{aligned}
\tilde{p}^*_{21,t} &= \frac{\tilde{P}^*_{21,t}}{e_{1,t}P_{22,t}}, \\
p^*_{21,t} &= \frac{P^*_{21,t}}{P_{22,t}}, \\
\tilde{p}^*_{12,t} &= \frac{e_{1,t}\tilde{P}^*_{12,t}}{P_{11,t}}, \\
p^*_{12,t} &= \frac{P^*_{12,t}}{P_{11,t}}.
\end{aligned}
$$

3. Inflation terms:

$$
\begin{aligned}
\pi_{12,t} &= \frac{P_{12,t}}{P_{12,t-1}} = \frac{q_{1,t}}{q_{1,t-1}}\pi_{11,t}, \\
\tilde{\pi}_{12,t} &= \frac{\tilde{P}_{12,t}}{\tilde{P}_{12,t-1}} = \frac{q_{1,t}}{q_{1,t-1}}\frac{\pi_{11,t}}{\pi_t^e}, \\
\pi_{21,t} &= \frac{P_{21,t}}{P_{21,t-1}} = \frac{q_{2,t-1}}{q_{2,t}}\pi_{22,t}, \\
\tilde{\pi}_{21,t} &= \frac{\tilde{P}_{21,t}}{\tilde{P}_{21,t-1}} = \frac{q_{2,t-1}}{q_{2,t}}\pi_{22,t}\pi_t^e, \\
\pi_{i,t}^c &= \frac{P_{i,t}^C/P_{ii,t}}{P_{i,t-1}^C/P_{ii,t-1}}\frac{P_{ii,t}}{P_{ii,t-1}} = \frac{\Phi_{i,t-1}}{\Phi_{i,t}}\pi_{ii,t}, \\
\pi_{i,t}^N &= \frac{P_{i,t}^N}{P_{i,t-1}^N} = \frac{q_{i,t}^N}{q_{i,t-1}^N}\pi_{ii,t}, \\
\pi_t^e &= \frac{e_{1,t}}{e_{1,t-1}} = \frac{rer_t}{rer_{t-1}}\frac{\pi_{1,t}^c}{\pi_{2,t}^c}.
\end{aligned}
$$

4. Price updating:

$$
\begin{aligned}
\bar{\pi}_{11}^{pT} &= \pi_{11}^{pT\gamma_{11}^{pT}}\pi^{*1-\gamma_{11}^{pT}}, \\
\bar{\pi}_{12}^{pT} &= \pi_{12}^{pT\gamma_{12}^{pT}}\pi^{*1-\gamma_{12}^{pT}}, \\
\bar{\pi}_{21}^{pT} &= \pi_{21}^{pT\gamma_{21}^{pT}}\pi^{*1-\gamma_{21}^{pT}}, \\
\bar{\pi}_{22}^{pT} &= \pi_{22}^{pT\gamma_{22}^{pT}}\pi^{*1-\gamma_{22}^{pT}}, \\
\bar{\pi}_{1}^{pN} &= \pi_{1}^{pN\gamma_{1}^{pN}}\pi^{*1-\gamma_{1}^{pN}}, \\
\bar{\pi}_{2}^{pN} &= \pi_{2}^{pN\gamma_{2}^{pN}}\pi^{*1-\gamma_{2}^{pN}}.
\end{aligned}
$$

5. Wages and wage inflation:

$$
\begin{aligned}
w_{i,t} &= \frac{W_{i,t}}{P_{ii,t}}, \\
w_{i,t}^* &= \frac{W_{i,t}^*}{P_{ii,t}}, \\
\omega_{i,t} &= \frac{W_{i,t}}{W_{i,t-1}} = \frac{w_{i,t}}{w_{i,t-1}}\pi_{ii,t}, \\
\bar{\omega}_{i,t+j} &= \omega_{i,t+j}^{\gamma_i^w}\pi^{*1-\gamma_i^w}.
\end{aligned}
$$

6. Real marginal costs:

$$mc^T_{i,t+j} = \frac{MC^T_{i,t+j}}{P_{ii,t}},$$
$$mc^N_{i,t+j} = \frac{MC^N_{i,t+j}}{P^N_{i,t+j}}.$$

7. Profits and assets:

$$pr_{i,t+j} = \frac{Pr_{i,t+j}}{P_{ii,t+j}},$$
$$b^G_{i,t+j} = \frac{B^G_{i,t+j}}{P^C_{i,t+j}},$$
$$b_{i,t+j} = \frac{B_{i,t+j}}{P^C_{i,t+j}},$$
$$b_{1,t+j} = \frac{B_{1,t+j}}{P^C_{1,t+j}},$$
$$b_{2,t+j} = \frac{B_{2,t+j}}{e_{1,t+j}P^C_{2,t+j}}.$$

D.2.2 Model equations

The following conditions need to be satisfied in the model with real and nominal rigidities.

1. Labor choice:

$$\frac{H^W_{i,t}}{G^W_{i,t}} = \left(\frac{W^*_{i,t}}{P_{ii,t}}\right)^{1-\frac{1+\theta^w_i}{\theta^w_i}\chi_i}$$

$$H^W_{i,t} = \chi_{0i}\left(\left[\frac{P_{ii,t}}{W_{i,t}}\right]^{-\frac{1+\theta^w_i}{\theta^w_i}} l_{i,t}\right)^{1-\chi_i} + \left(\beta_i\xi^w_i\right)^1 \mathbb{E}_t\left(\frac{\bar{\omega}_{i,t}}{\pi_{ii,t+1}}\right)^{-\frac{1+\theta^w_i}{\theta^w_i}(1-\chi_i)} H^W_{i,t+1}$$

$$G^W_{i,t} = \frac{1-\tau^w_i}{1+\theta^w_i}\lambda_{i,t}P^C_{i,t}\Phi_{i,t}\left[\frac{P_{ii,t}}{W_{i,t}}\right]^{-\frac{1+\theta^w_i}{\theta^w_i}} l_{i,t} + \beta_i\xi^w_i\mathbb{E}_t\left(\frac{\bar{\omega}_{i,t}}{\pi_{ii,t+1}}\right)^{\frac{-1}{\theta^w_i}} G^W_{i,t+1}$$

2. Aggregate wage evolution

$$\frac{W_{i,t+j}}{P_{ii,t+j}} = \left[(1-\xi^w_i)\left(\frac{W^*_{i,t+j}}{P_{ii,t+j}}\right)^{\frac{-1}{\theta^w_i}} + \xi^w_i\left(\frac{\bar{\omega}_{i,t+j-1}}{\pi_{ii,t+j}}\frac{W_{i,t+j-1}}{P_{ii,t+j-1}}\right)^{\frac{-1}{\theta^w_i}}\right]^{-\theta^w_i}$$

3. Marginal utility of consumption

$$\left(c_{i,t+j}\left(h\right) - \kappa^c_i c_{i,t+j-1} - \nu^c_{i,t+j}\right)^{-\sigma} = \lambda_{i,t+j}\left(h\right)P^C_{i,t+j}$$

4. Investment choice

$$\begin{aligned} 0 &= -\lambda_{i,t+j}\left(h\right)P^{C}_{i,t+j}\frac{\Phi_{i,t+j}}{\Pi_{i,t+j}} - \lambda_{i,t+j}P^{C}_{i,t+j}\frac{\Phi_{i,t+j}}{\Pi_{i,t+j}}\phi^{I}_{i}\frac{\left(x_{i,t+j}\left(h\right) - x_{i,t-1+j}\left(h\right)\right)}{x_{i,t-1+j}\left(h\right)} \\ &\quad -\beta_{i}\lambda_{i,t+1+j}\left(h\right)P^{C}_{i,t+1+j}\frac{\Phi_{i,t+1+j}}{\Pi_{i,t+1+j}}\frac{1}{2}\phi^{I}_{i}\frac{x^{2}_{i,t+j}\left(h\right) - x^{2}_{i,t+1+j}\left(h\right)}{x^{2}_{i,t+j}\left(h\right)} \\ &\quad +\mu_{i,t+j}\left(h\right) \end{aligned}$$

5. Capital choice

$$E_{t}\beta_{i}\lambda_{i,t+1+j}\left(h\right)P^{C}_{i,t+1+j}\Phi_{i,t+1+j}\left(1-\tau^{R}_{i,t+1+j}\right)\frac{R_{i,t+1+j}}{P_{ii,t+1+j}} + E_{t}\beta_{i}\mu_{i,t+1+j}\left(h\right)\left(1-\delta\right) = \mu_{i,t+j}\left(h\right)$$

Absent investment adjustment costs, i.e., $\phi^{I}_{i} = 0$, it is $\lambda_{i,t+j}\left(h\right)P^{C}_{i,t+j}\frac{\Phi_{i,t+j}}{\Pi_{i,t+j}} = \mu_{i,t+j}\left(h\right)$ and thus

$$E_{t}\beta\frac{\lambda_{i,t+1+j}\left(h\right)P^{C}_{i,t+1+j}}{\lambda_{i,t+j}\left(h\right)P^{C}_{i,t+j}}\frac{\Phi_{i,t+1+j}}{\Phi_{i,t+j}}\frac{\Pi_{i,t+j}}{\Pi_{i,t+1+j}}\left\{\left(1-\tau^{R}_{i,t+1+j}\right)\Pi_{i,t+1+j}\frac{R_{i,t+1+j}}{P_{ii,t+1+j}} + \left(1-\delta\right)\right\} = 1.$$

6. Capital accumulation

$$k_{i,t+j} \leq \left(1-\delta\right)k_{i,t-1+j} + x_{i,t+j}$$

7. Consumption budget constraint 1

$$\begin{aligned} c_{1,t+j} &= \Phi_{1,t+j}\left(\left(1-\tau^{W}_{1,t+j}\right)w_{1,t+j}l_{1,t+j} + \left(1-\tau^{R}_{1,t+j}\right)r_{1,t+j}k_{1i,t-1+j}\right) \\ &\quad -\frac{\Phi_{1,t+j}}{\Pi_{1,t+j}}\left(x_{1,t+j} + \frac{1}{2}\phi^{I}_{1}\frac{\left(x_{1,t+j} - x_{1,t-1+j}\right)^{2}}{x_{1,t-1+j}}\right) \\ &\quad +\Phi_{1,t+j}pr_{1,t+j} \\ &\quad -\Phi_{1,t+j}\left(Q^{G}_{1,t+j}b^{G}_{1,t+j} - b^{G}_{1,t-1+j}\frac{1}{\pi_{11,t+j}}\right) \\ &\quad -\left(Q^{B}_{1,t+j}b_{1,t+j} - b_{1,t-1+j}\frac{1}{\pi^{c}_{1,t+j}}\right) \\ &\quad -\Gamma\left(\frac{B_{1,t+j}\left(h\right)}{P^{c}_{1,t+j}}\right) + \Phi_{1,t+j}tr_{1,t+j} \end{aligned}$$

$$
\begin{aligned}
c_{1,t+j} \quad = \quad & -\frac{\Phi_{1,t+j}}{\Pi_{1,t+j}}\left(x_{1,t+j}+\frac{1}{2}\phi_{1}^{I}\frac{\left(x_{1,t+j}-x_{1,t-1+j}\right)^{2}}{x_{1,t-1+j}}\right) \\
& -\Phi_{1,t+j}\left(Q_{1,t+j}^{G}b_{1,t+j}^{G}-b_{1,t-1+j}^{G}\frac{1}{\pi_{11,t+j}}\right) \\
& -\left(Q_{1,t+j}^{B}b_{1,t+j}-b_{1,t-1+j}\frac{1}{\pi_{1,t+j}^{c}}\right) \\
& +\Phi_{1,t}\left(c_{11,t}+x_{11,t}\right) \\
& +\Phi_{1,t}\frac{q_{t}}{q_{2,t}}\left(c_{21,t}+x_{21,t}\right)\zeta_{2} \\
& +\Phi_{1,t}q_{1,t}^{N}\left(c_{1,t}^{N}+\eta\left(c_{11,t}+c_{12,t}\right)\right)
\end{aligned}
$$

for country 2

$$
\begin{aligned}
c_{2,t+j} \quad = \quad & \Phi_{2,t+j}\left(\left(1-\tau_{2,t+j}^{W}\right)w_{2,t+j}l_{2,t+j}+\left(1-\tau_{2,t+j}^{R}\right)r_{2,t+j}k_{2,t-1+j}\right) \\
& -\frac{\Phi_{2,t+j}}{\Pi_{2,t+j}}\left(x_{2,t+j}-\frac{1}{2}\phi_{2}^{I}\frac{\left(x_{2,t+j}-x_{2,t-1+j}\right)^{2}}{x_{2,t-1+j}}\right) \\
& +\Phi_{2,t+j}pr_{2,t+j} \\
& -\Phi_{2,t+j}\left(Q_{2,t+j}^{G}b_{2,t+j}^{G}-b_{2,t-1+j}^{G}\frac{1}{\pi_{22,t+j}}\right) \\
& -\left(Q_{2,t+j}^{B}b_{2,t+j}-b_{2,t-1+j}\frac{1}{\pi_{1,t+j}^{e}\pi_{2,t+j}^{c}}\right) \\
& -\Gamma\left(\frac{B_{2,t+j}\left(h\right)}{e_{1,t+j}P_{2,t+j}^{C}}\right)+\Phi_{2,t+j}tr_{2,t+j}
\end{aligned}
$$

$$
\begin{aligned}
c_{2,t+j} \quad = \quad & -\frac{\Phi_{2,t+j}}{\Pi_{2,t+j}}\left(x_{2,t+j}-\frac{1}{2}\phi_{2}^{I}\frac{\left(x_{2,t+j}-x_{2,t-1+j}\right)^{2}}{x_{2,t-1+j}}\right) \\
& -\Phi_{2,t+j}\left(Q_{2,t+j}^{G}b_{2,t+j}^{G}-b_{2,t-1+j}^{G}\frac{1}{\pi_{22,t+j}}\right) \\
& -\Phi_{2,t+j}\left(Q_{2,t+j}^{B}b_{2,t+j}-b_{2,t-1+j}\frac{1}{\pi_{1,t+j}^{e}\pi_{22,t+j}}\right) \\
& +\Phi_{2,t}\left(c_{22,t}+x_{22,t}\right) \\
& +\Phi_{2,t}\frac{q_{2,t}}{q_{t}}\left(c_{12,t}+x_{12,t}\right)\frac{1}{\zeta_{2}} \\
& +\Phi_{2,t}q_{2,t}^{N}\left(c_{2,t}^{N}+\eta\left(c_{22,t}+c_{21,t}\right)\right)
\end{aligned}
$$

and

$$b_{2,t} = \frac{1}{\zeta_2}\frac{1}{rer_t}b_{1,t}$$

8. Risk sharing condition: with adjustment costs

$$\left(\beta_1 \frac{\lambda_{1,t+1+j}P^C_{1,t+1+j}}{\lambda_{1,t+j}P^C_{1,t+j}} - \beta_2 \frac{\lambda_{2,t+1+j}P^C_{2,t+1+j}}{\lambda_{2,t+j}P^C_{2,t+j}} \frac{rer_{1,t+j}}{rer_{1,t+1+j}}\right) \frac{1}{\pi^c_{1,t+1+j}} = \Gamma'\left(b_{1,t+j}\right) - \Gamma'\left(b_{2,t+j}\right)$$

9. Price of bond derived from country 1

$$\begin{aligned} Q_{1,t+j} &= \beta_1 \frac{\lambda_{1,t+1+j}}{\lambda_{1,t+j}} - \Gamma'\left(\frac{B_{1,t+j}}{P^C_{1,t+j}}\right) \\ &= \beta_1 \frac{\lambda_{1,t+1+j}P^C_{1,t+1+j}}{\lambda_{1,t+j}P^C_{1,t+j}} \frac{1}{\pi^c_{1,t+j}} - \Gamma'\left(b_{1,t+j}\right) \\ Q_{2,t+j} &= \beta_2 \frac{\lambda_{2,t+1+j}P^C_{2,t+1+j}}{\lambda_{2,t+j}P^C_{2,t+j}} \frac{rer_{t+j}}{rer_{t+1+j}} \frac{1}{\pi^c_{1,t+j}} - \Gamma'\left(b_{2,t+j}\right) \end{aligned}$$

10. Bond market clearing

$$b_{1,t+j} + \nu_2 b_{2,t+j} q_{t+j} = 0$$

11. Investment aggregate: for country i

$$x_{i,t+j} = \left[\left(\alpha^I_{i1}\right)^{\frac{1}{\varepsilon^I_i}} \left(x_{i1,t+j}\right)^{\frac{\varepsilon^I_i - 1}{\varepsilon^I_i}} + \left(\alpha^I_{i2}\right)^{\frac{1}{\varepsilon^I_i}} \left(x_{i2,t+j}\right)^{\frac{\varepsilon^I_i - 1}{\varepsilon^I_i}}\right]^{\frac{\varepsilon^I_i}{\varepsilon^I_i - 1}}$$

12. Investment optimal choices:

$$\begin{aligned} x_{12,t+j} &= \frac{\alpha^I_{12}}{\alpha^I_{11}} \left(\frac{1}{q_{1,t+j}}\right)^{\varepsilon^I_i} x_{11,t+j} \\ x_{22,t+j} &= \frac{\alpha^I_{22}}{\alpha^I_{21}} \left(\frac{1}{q_{2,t+j}}\right)^{\varepsilon^I_i} x_{21,t+j} \end{aligned}$$

13. Consumption aggregate over traded goods: for country i

$$c^T_{i,t+j} = \left[\left(\alpha^T_{i1}\right)^{\frac{1}{\varepsilon^T_i}} \left(c_{i1,t+j}\right)^{\frac{\varepsilon^T_i - 1}{\varepsilon^T_i}} + \left(\alpha^T_{i2}\right)^{\frac{1}{\varepsilon^T_i}} \left(c_{i2,t+j}\right)^{\frac{\varepsilon^T_i - 1}{\varepsilon^T_i}}\right]^{\frac{\varepsilon^T_i}{\varepsilon^T_i - 1}}$$

14. Consumption aggregate over traded and non traded goods: for country i

$$c_{i,t+j} = \left[\left(\alpha^N_{i1}\right)^{\frac{1}{\varepsilon^N_i}} \left(c^T_{i,t+j}\right)^{\frac{\varepsilon^N_i - 1}{\varepsilon^N_i}} + \left(\alpha^N_{i2}\right)^{\frac{1}{\varepsilon^N_i}} \left(c^N_{i,t+j}\right)^{\frac{\varepsilon^N_i - 1}{\varepsilon^N_i}}\right]^{\frac{\varepsilon^N_i}{\varepsilon^N_i - 1}}$$

15. Consumption optimal choices non-traded goods: for country 1

$$c^N_{1,t+j} = c_{12,t+j}\frac{\alpha^N_{12}}{\alpha^N_{11}}\left[\left(\alpha^T_{11}\right)^{\frac{1}{\varepsilon^T_i}}\left(\frac{c_{11,t+j}}{c_{12,t+j}}\right)^{\frac{\varepsilon^T_i-1}{\varepsilon^T_i}} + \left(\alpha^T_{12}\right)^{\frac{1}{\varepsilon^T_i}}\right]^{\frac{\varepsilon^T_i-\varepsilon^N_i}{\varepsilon^T_i-1}}\left(\frac{q^N_{1,t+j}}{q_{1,t+j}+\eta q^N_{1,t+j}}\right)^{-\varepsilon^N_i}\left(\frac{1}{\alpha^T_{12}}\right)^{\frac{\varepsilon^N_i}{\varepsilon^T_i}}$$

and for country 2

$$c^N_{2,t+j} = c_{21,t+j}\frac{\alpha^N_{22}}{\alpha^N_{21}}\left[\left(\alpha^T_{21}\right)^{\frac{1}{\varepsilon^T_i}} + \left(\alpha^T_{22}\right)^{\frac{1}{\varepsilon^T_i}}\left(\frac{c_{22,t+j}}{c_{21,t+j}}\right)^{\frac{\varepsilon^T_i-1}{\varepsilon^T_i}}\right]^{\frac{\varepsilon^T_i-\varepsilon^N_i}{\varepsilon^T_i-1}}\left(\frac{q^N_{2,t+j}}{\frac{1}{q_{2,t+j}}+\eta q^N_{2,t+j}}\right)^{-\varepsilon^N_i}\left(\frac{1}{\alpha^T_{21}}\right)^{\frac{\varepsilon^N_i}{\varepsilon^T_i}}$$

16. Consumption optimal choice traded goods: for country 1

$$c_{12,t+j} = \frac{\alpha^T_{12}}{\alpha^T_{11}}\left(\frac{1+\eta q^N_{1,t+j}}{q_{1,t+j}+\eta q^N_{1,t+j}}\right)^{\varepsilon^T_1} c_{11,t+j}$$

for country 2

$$c_{22,t+j} = \frac{\alpha^T_{22}}{\alpha^T_{21}}\left(\frac{\frac{1}{q_{2,t+j}}+\eta q^N_{2,t+j}}{1+\eta q^N_{2,t+j}}\right)^{\varepsilon^T_2} c_{21,t+j}$$

17. Price definition Φ_i: for country 1

$$\Phi_{1,t+j} = \frac{\frac{c_{1,t+j}}{c^N_{1,t+j}}}{\left[\left(1+\eta q^N_{1,t+j}\right)\frac{c_{11,t+j}}{c_{12,t+j}} + \left(q_{1,t+j}+\eta q^N_{1,t+j}\right)\right]\frac{c_{12,t+j}}{c^N_{1,t+j}} + q^N_{1,t+j}}$$

for country 2

$$\Phi_{2,t+j} = \frac{\frac{c_{2,t+j}}{c^N_{2,t+j}}}{\left[\left(\frac{1}{q_{2,t+j}}+\eta q^N_{2,t+j}\right)\frac{c_{21,t+j}}{c_{22,t+j}} + \left(1+\eta q^N_{2,t+j}\right)\right]\frac{c_{22,t+j}}{c^N_{2,t+j}} + q^N_{2,t+j}}$$

18. Price definition Π_i: for country 1

$$\Pi_{1,t+j} = \frac{P_{1,t+j}}{P^I_{1,t+j}} = \frac{1}{\frac{x^T_{11,t+j}}{x^T_{i2,t+j}} + q_{1,t+j}}\frac{x_{1,t+j}}{x_{12,t+j}}$$

for country 2

$$\Pi_{2,t+j} = \frac{P_{2,t+j}}{P^I_{2,t+j}} = \frac{1}{\frac{x^T_{21,t+j}}{x^T_{22,t+j}} + \frac{1}{q_{2,t+j}}}\frac{x_{2,t+j}}{x_{22,t+j}}$$

19. Consumption real exchange rate

$$
\begin{aligned}
rer_{t+j} &= \frac{P^C_{2,t+j}e_{1,t+j}}{P^C_{1,t+j}} = \frac{\Phi_{1,t+j}}{\Phi_{2,t+j}}q_{t+j} \\
q_{t+j} &= \frac{e_{1,t+j}P_{22,t+j}}{P_{11,t+j}}
\end{aligned}
$$

20. Production traded goods: for country i

$$
y^T_{i,t+j} = \left[\left(\omega^T_{li}\right)^{1-\kappa^T_i}\left(A^T_{i,t+j}l^T_{i,t+j}\right)^{\kappa^T_i} + \left(\omega^T_{ki}\right)^{1-\kappa^T_i}\left(k^T_{i,t+j}\right)^{\kappa^T_i}\right]^{\frac{1}{\kappa^T_i}}
$$

21. Production non-traded goods: for country i

$$
y^N_{i,t+j} = \left[\left(\omega^N_{li}\right)^{1-\kappa^N_i}\left(A^N_{i,t+j}l^N_{i,t+j}\right)^{\kappa^N_i} + \left(\omega^N_{ki}\right)^{1-\kappa^N_i}\left(k^N_{i,t+j}\right)^{\kappa^N_i}\right]^{\frac{1}{\kappa^N_i}}
$$

22. Optimal labor input choices traded goods: for country i

$$
w_{i,t+j} = mc^T_{i,t+j}A^T_{i,t+j}\left(\frac{\omega^T_{li}y^T_{i,t+j}}{A^T_{i,t+j}l^T_{i,t+j}}\right)^{1-\kappa^T_i}
$$

23. Optimal capital input choices traded goods: for country i

$$
r_{i,t+j} = mc^T_{i,t+j}\left(\frac{\omega^T_{ki}y^T_{i,t+j}}{k^T_{i,t+j}}\right)^{1-\kappa^T_i}
$$

24. Optimal labor input choices traded goods: for country i

$$
w_{i,t+j} = q^N_{i,t+j}mc^N_{i,t+j}A^N_{i,t+j}\left(\frac{\omega^N_{li}y^N_{i,t+j}}{A^N_{i,t+j}l^N_{i,t+j}}\right)^{1-\kappa^N_i}
$$

25. Optimal capital input choices traded goods: for country i

$$
r_{i,t+j} = q^N_{i,t+j}mc^N_{i,t+j}\left(\frac{\omega^N_{ki}y^N_{i,t+j}}{k^N_{i,t+j}}\right)^{1-\kappa^N_i}
$$

26. Labor market clearing

$$
l^T_{i,t+j} + l^N_{i,t+j} = l_{i,t+j}
$$

27. Capital market clearing

$$
k^T_{i,t+j} + k^N_{i,t+j} = k_{i,t-1+j}
$$

28. Goods market clearing for good 1:

$$y_1^T = \Delta_{11,t}\left[c_{11,t} + x_{11,t}\right] + \Delta_{21,t}\left[c_{21,t} + x_{21,t}\right]\nu_2$$

29. Evolution of dispersion index good 1: country 1

$$\Delta_{11,t}^T = \left(1-\xi_{11}^{pT}\right)\left(p_{11,t}^*\right)^{\frac{-\left(1+\theta_1^{pT}\right)}{\theta_1^{pT}}} + \xi_{11}^{pT}\left(\frac{\bar{\pi}_{11,t-1}}{\pi_{11,t}}\right)^{\frac{-\left(1+\theta_1^{pT}\right)}{\theta_1^{pT}}}\Delta_{11,t-1}^T$$

country 2 under producer currency pricing

$$\Delta_{21,t}^T = \left(1-\xi_{21}^{pT}\right)\left(\tilde{p}_{21,t}^* q_{2,t}\right)^{-\frac{1+\theta_1^{pT}}{\theta_1^{pT}}} + \xi_{21}^{pT}\left(\frac{\bar{\tilde{\pi}}_{21,t-1}}{\tilde{\pi}_{21,t}}\right)^{-\frac{1+\theta_1^{pT}}{\theta_1^{pT}}}\Delta_{21,t-1}^T$$

$$\tilde{\pi}_{21,t} = \frac{q_{2,t-1}}{q_{2,t}}\pi_{22,t}\pi_{1,t}^e$$

under local currency pricing

$$\Delta_{21,t}^T = \left(1-\xi_{21}^{pT}\right)\left(p_{21,t}^* q_{2,t}\right)^{-\frac{1+\theta_1^{pT}}{\theta_1^{pT}}} + \xi_{21}^{pT}\left(\frac{\bar{\pi}_{21,t-1}}{\pi_{21,t}}\right)^{-\frac{1+\theta_1^{pT}}{\theta_1^{pT}}}\Delta_{21,t-1}^T$$

30. Goods market clearing for good 2:

$$\nu_2 y_2^T = \Delta_{12,t}\left[c_{12,t} + x_{12,t}\right] + \Delta_{22,t}\left[c_{22,t} + x_{22,t}\right]\nu_2$$

31. Evolution of dispersion index good 2: country 2

$$\Delta_{22,t}^T = \left(1-\xi_{22}^{pT}\right)\left(p_{22,t}^*\right)^{\frac{-\left(1+\theta_2^{pT}\right)}{\theta_2^{pT}}} + \xi_{22}^{pT}\left(\frac{\bar{\pi}_{22,t-1}}{\pi_{22,t}}\right)^{\frac{-\left(1+\theta_2^{pT}\right)}{\theta_2^{pT}}}\Delta_{22,t-1}^T$$

country 1 under producer currency pricing

$$\Delta_{12,t}^T = \left(1-\xi_{12}^{pT}\right)\left(\frac{\tilde{p}_{12,t}^*}{q_{1,t}}\right)^{-\frac{1+\theta_2^{pT}}{\theta_2^{pT}}} + \xi_{12}^{pT}\left(\frac{\bar{\tilde{\pi}}_{12,t-1}}{\tilde{\pi}_{12,t}}\right)^{-\frac{1+\theta_2^{pT}}{\theta_2^{pT}}}\Delta_{12,t-1}^T$$

$$\tilde{\pi}_{12,t} = \frac{q_{1,t}}{q_{1,t-1}}\frac{\pi_{11,t}}{\pi_{1,t}^e}$$

under local currency pricing

$$\Delta_{12,t}^T = \left(1-\xi_{12}^{pT}\right)\left(\frac{p_{12,t}^*}{q_{1,t}}\right)^{-\frac{1+\theta_2^{pT}}{\theta_2^{pT}}} + \xi_{12}^{pT}\left(\frac{\bar{\pi}_{12,t-1}}{\pi_{12,t}}\right)^{-\frac{1+\theta_2^{pT}}{\theta_2^{pT}}}\Delta_{12,t-1}^T$$

32. Goods market clearing non-traded goods: for country i

$$y^N_{i,t+j} = \Delta^N_{i,t+j}\left[c^N_{i,t+j} + \eta\left(c_{i1,t+j} + c_{i2,t+j}\right)\right]$$

33. Evolution of dispersion index non-traded good i:

$$\begin{aligned}
\Delta^N_{i,t} &= \left(1-\xi^{pN}_i\right)\left(\frac{q^{N*}_{i,t}}{q^N_{i,t}}\right)^{\frac{-\left(1+\theta^{pN}_i\right)}{\theta^{pN}_i}} + \xi^{pN}_i\left(\frac{\bar{\pi}^N_{i,t-1}}{\pi^N_{i,t}}\right)^{\frac{-\left(1+\theta^{pN}_i\right)}{\theta^{pN}_i}}\Delta^N_{i,t-1} \\
\pi^N_{i,t} &= \frac{q^N_{i,t}}{q^N_{i,t-1}}\pi_{ii,t}
\end{aligned}$$

34. Price indices good 1: country 1

$$1 = \left(1-\xi^{pT}_{11}\right)\left(p^*_{11,t}\right)^{-\frac{1}{\theta^{pT}_1}} + \xi^{pT}_{11}\left(\frac{\bar{\pi}_{11,t-1}}{\pi_{11,t}}\right)^{-\frac{1}{\theta^{pT}_1}}$$

for country 2: if producer currency pricing

$$1 = \left(1-\xi^{pT}_{21}\right)\left(\tilde{p}^*_{21,t}q_{2,t}\right)^{-\frac{1}{\theta^{pT}_1}} + \xi^{pT}_{21}\left(\frac{\bar{\tilde{\pi}}_{21,t-1}}{\tilde{\pi}_{21,t}}\right)^{-\frac{1}{\theta^{pT}_1}}$$

if local currency pricing

$$1 = \left(1-\xi^{pT}_{21}\right)\left(p^*_{21,t}q_{2,t}\right)^{-\frac{1}{\theta^{pT}_1}} + \xi^{pT}_{21}\left(\frac{\bar{\pi}_{21,t-1}}{\pi_{21,t}}\right)^{-\frac{1}{\theta^{pT}_1}}$$

35. Price indices good 2: country 2

$$1 = \left(1-\xi^{pT}_{22}\right)\left(p^*_{22,t}\right)^{-\frac{1}{\theta^{pT}_2}} + \xi^{pT}_{22}\left(\frac{\bar{\pi}_{22,t-1}}{\pi_{22,t}}\right)^{-\frac{1}{\theta^{pT}_2}}$$

for country 1: if producer currency pricing

$$1 = \left(1-\xi^{pT}_{12}\right)\left(\frac{\tilde{p}^*_{12,t}}{q_{1,t}}\right)^{-\frac{1}{\theta^{pT}_2}} + \xi^{pT}_{12}\left(\frac{\bar{\tilde{\pi}}_{12,t-1}}{\tilde{\pi}_{12,t}}\right)^{-\frac{1}{\theta^{pT}_2}}$$

if local currency pricing

$$1 = \left(1-\xi^{pT}_{12}\right)\left(\frac{p^*_{12,t}}{q_{1,t}}\right)^{-\frac{1}{\theta^{pT}_2}} + \xi^{pT}_{12}\left(\frac{\bar{\pi}_{12,t-1}}{\pi_{12,t}}\right)^{-\frac{1}{\theta^{pT}_2}}$$

36. Price index non-traded good

$$1 = \left(1-\xi^{pN}_i\right)\left(\frac{q^{N*}_{i,t}}{q^N_{i,t}}\right)^{\frac{-1}{\theta^{pN}_i}} + \xi^{pN}_i\left(\frac{\bar{\pi}^N_{i,t-1}}{\pi^N_{i,t}}\right)^{\frac{-1}{\theta^{pN}_i}}$$

37. Stochastic discount factors

$$\psi_{1,t,t+1} = \beta_1 \frac{\lambda_{1,t+1} P^C_{1,t+1}}{\lambda_{1,t} P^C_{1,t}} \frac{P^C_{1,t}}{P^C_{1,t+1}}$$
$$\psi_{2,t,t+1} = \beta_2 \frac{\lambda_{2,t+1} P^C_{2,t+1}}{\lambda_{2,t} P^C_{2,t}} \frac{P^C_{2,t}}{P^C_{2,t+1}}$$

38. Price for traded good country 1

$$\frac{P^*_{11,t}}{P_{11,t}} = \frac{H^{pT}_{11,t}}{G^{pT}_{11,t}},$$
$$H^{pT}_{11,t} = \frac{MC_{1,t}}{P_{11,t}} y^T_{11,t} + \xi^{pT}_{11} E_t \left\{ \psi_{1,t,t+1} \pi_{11,t+1} \left(\frac{\bar{\pi}_{11,t}}{\pi_{11,t+1}} \right)^{\frac{-\left(1+\theta^{pT}_1\right)}{\theta^{pT}_1}} H^{pT}_{11,t+1} \right\}$$
$$G^{pT}_{11,t} = \frac{1+\tau_{11}}{1+\theta^{pT}_1} y^T_{11,t} + \xi^{pT}_{11} E_t \left\{ \psi_{1,t,t+1} \pi_{11,t+1} \left(\frac{\bar{\pi}_{11,t}}{\pi_{11,t+1}} \right)^{1-\frac{\left(1+\theta^{pT}_1\right)}{\theta^{pT}_1}} G^{pT}_{11,t+1} \right\}$$

39. Export price for traded good country 1: producer currency pricing

$$\frac{\tilde{P}^*_{21,t}}{\tilde{P}_{21,t}} = \frac{\tilde{H}^{pT}_{21,t}}{\tilde{G}^{pT}_{21,t}}$$
$$\tilde{H}^{pT}_{21,t} = \frac{MC_{1,t}}{P_{11,t}} \frac{P_{11,t}}{e_{1,t} P_{21,t}} y^T_{21,t} + \xi^{pT}_{21} E_t \left\{ \psi_{1,t,t+1} \tilde{\pi}_{21,t+1} \left(\frac{\bar{\tilde{\pi}}_{21,t}}{\tilde{\pi}_{21,t+1}} \right)^{\frac{-(1+\theta^{pT}_1)}{\theta^{pT}_1}} \tilde{H}^{pT}_{21,t+1} \right\}$$
$$\tilde{G}^{pT}_{21,t} = \frac{1+\tau_{21}}{1+\theta^{pT}_1} y^T_{21,t} + \xi^{pT}_{21} E_t \left\{ \psi_{1,t,t+1} \tilde{\pi}_{21,t+1} \left(\frac{\bar{\tilde{\pi}}_{21,t}}{\tilde{\pi}_{21,t+1}} \right)^{1-\frac{(1+\theta^{pT}_1)}{\theta^{pT}_1}} \tilde{G}^{pT}_{21,t+1} \right\}$$

local currency pricing

$$\frac{P^*_{21,t}}{P_{21,t}} = \frac{H^{pT}_{21,t}}{G^{pT}_{21,t}}$$
$$H^{pT}_{21,t} = \frac{MC_{1,t}}{P_{11,t}} \frac{P_{11,t}}{e_{1,t} P_{21,t}} y^T_{21,t} + \xi^{pT}_{21} E_t \left\{ \psi_{1,t,t+1} \pi_{21,t+1} \left(\frac{\bar{\pi}_{21,t}}{\pi_{21,t+1}} \right)^{\frac{-(1+\theta^{pT}_1)}{\theta^{pT}_1}} H^{pT}_{21,t+1} \right\}$$
$$G^{pT}_{21,t} = \frac{1+\tau_{21}}{1+\theta^{pT}_1} y^T_{21,t} + \xi^{pT}_{21} E_t \left\{ \psi_{1,t,t+1} \pi_{21,t+1} \left(\frac{\bar{\pi}_{21,t}}{\pi_{21,t+1}} \right)^{1-\frac{(1+\theta^{pT}_1)}{\theta^{pT}_1}} G^{pT}_{21,t+1} \right\}$$

40. Price for traded good country 2

$$
\begin{aligned}
\frac{P^*_{22,t}}{P_{22,t}} &= \frac{H^{pT}_{22,t}}{G^{pT}_{22,t}}, \\
H^{pT}_{22,t} &= \frac{MC_{2,t}}{P_{22,t}} y^T_{22,t} + \xi^{pT}_{22} E_t \left\{ \psi_{2,t,t+1} \pi_{22,t+1} \left(\frac{\bar{\pi}_{22,t}}{\pi_{22,t+1}} \right)^{\frac{-\left(1+\theta^{pT}_1\right)}{\theta^{pT}_2}} H^{pT}_{22,t+1} \right\} \\
G^{pT}_{22,t} &= \frac{1+\tau_{22}}{1+\theta^{pT}_2} y^T_{22,t} + \xi^{pT}_{22} E_t \left\{ \psi_{2,t,t+1} \pi_{22,t+1} \left(\frac{\bar{\pi}_{22,t}}{\pi_{22,t+1}} \right)^{1-\frac{\left(1+\theta^{pT}_1\right)}{\theta^{pT}_2}} G^{pT}_{22,t+1} \right\}
\end{aligned}
$$

41. Export price for traded good country 2: producer currency pricing

$$
\begin{aligned}
\frac{\tilde{P}^*_{12,t}}{\tilde{P}_{12,t}} &= \frac{\tilde{H}^{pT}_{12,t}}{\tilde{G}^{pT}_{12,t}} \\
\tilde{H}^{pT}_{12,t} &= \frac{MC_{2,t}}{P_{22,t}} \frac{e_{1,t} P_{22,t}}{P_{12,t}} y^T_{12,t} + \xi^{pT}_{12} E_t \left\{ \psi_{2,t,t+1} \tilde{\pi}_{12,t+1} \left(\frac{\bar{\tilde{\pi}}_{12,t}}{\tilde{\pi}_{12,t+1}} \right)^{\frac{-(1+\theta^{pT}_2)}{\theta^{pT}_2}} \tilde{H}^{pT}_{12,t+1} \right\} \\
\tilde{G}^{pT}_{12,t} &= \frac{1+\tau_{12}}{1+\theta^{pT}_2} y^T_{12,t} + \xi^{pT}_{12} E_t \left\{ \psi_{2,t,t+1} \tilde{\pi}_{12,t+1} \left(\frac{\bar{\tilde{\pi}}_{12,t}}{\tilde{\pi}_{12,t+1}} \right)^{1-\frac{(1+\theta^{pT}_2)}{\theta^{pT}_2}} \tilde{G}^{pT}_{12,t+1} \right\}
\end{aligned}
$$

local currency pricing

$$
\begin{aligned}
\frac{P^*_{12,t}}{P_{12,t}} &= \frac{H^{pT}_{12,t}}{G^{pT}_{12,t}} \\
H^{pT}_{12,t} &= \frac{MC_{2,t}}{P_{22,t}} \frac{e_{1,t} P_{22,t}}{P_{12,t}} y^T_{12,t} + \xi^{pT}_{12} E_t \left\{ \psi_{2,t,t+1} \pi_{12,t+1} \left(\frac{\bar{\pi}_{12,t}}{\pi_{12,t+1}} \right)^{\frac{-(1+\theta^{pT}_2)}{\theta^{pT}_2}} H^{pT}_{12,t+1} \right\} \\
G^{pT}_{12,t} &= \frac{1+\tau_{12}}{1+\theta^{pT}_2} y^T_{12,t} + \xi^{pT}_{12} E_t \left\{ \psi_{2,t,t+1} \pi_{12,t+1} \left(\frac{\bar{\pi}_{12,t}}{\pi_{12,t+1}} \right)^{1-\frac{(1+\theta^{pT}_2)}{\theta^{pT}_2}} G^{pT}_{12,t+1} \right\}
\end{aligned}
$$

42. Price for non-traded good country i:

$$\begin{aligned}
\frac{P_{i,t}^{N*}}{P_{i,t}^{N}} &= \frac{H_{i,t}^{pN}}{G_{i,t}^{pN}}, \\
H_{i,t}^{pN} &= \frac{MC_{i,t}}{P_{ii,t}}\frac{P_{ii,t}}{P_{i,t}^{N}} y_{i,t}^{N} + \xi_i^{pN} E_t \left\{ \psi_{i,t,t+1} \pi_{i,t+1}^{N} \left(\frac{\bar{\pi}_{i,t}^{N}}{\pi_{i,t+1}^{N}} \right)^{\frac{-\left(1+\theta_i^{pN}\right)}{\theta_i^{pN}}} H_{i,t+1}^{pN} \right\} \\
G_{i,t}^{pN} &= \frac{1+\tau_i^{N}}{1+\theta_i^{pN}} y_{i,t}^{N} + \xi_i^{pN} E_t \left\{ \psi_{i,t,t+1} \pi_{i,t+1}^{N} \left(\frac{\bar{\pi}_{i,t}^{N}}{\pi_{i,t+1}^{N}} \right)^{1-\frac{\left(1+\theta_i^{pN}\right)}{\theta_i^{pN}}} G_{i,t+1}^{pN} \right\}
\end{aligned}$$

43. Definition of real profits $\frac{Pr_{i,t+j}}{P_{ii,t+j}}$: profits from home activities

$$\begin{aligned}
Pr_{11,t} &= \int_0^1 \left((1+\tau_{11}) P_{11,t}(h) - MC_{1,t} \right) \left(c_{11,t}(h) + x_{11,t}(h) \right) dh \\
pr_{11,t} &= \left[(1+\tau_{11}) - mc_{1,t} \Delta_{11,t}^{T} \right] \left(c_{11,t} + x_{11,t} \right)
\end{aligned}$$

from exporting with producer currency pricing

$$\begin{aligned}
Pr_{21,t} &= \int_0^{\bar{h}_1} \left((1+\tau_{21}) \tilde{P}_{21,t}(h) - MC_{1,t} \right) \left(c_{21,t}(h) + x_{21,t}(h) \right) \nu_2 dh \\
\frac{Pr_{21,t}}{P_{11,t}} &= \left[(1+\tau_{21}) \frac{q_t}{q_{2,t}} - mc_{1,t} \Delta_{21,t}^{T} \right] \left(c_{21,t} + x_{21,t} \right) \nu_2
\end{aligned}$$

with local currency pricing

$$\begin{aligned}
Pr_{21,t} &= \int_0^{\bar{h}_1} \left((1+\tau_{21}) e_{1,t} P_{21,t}(h) - MC_{1,t} \right) \left(c_{21,t}(h) + x_{21,t}(h) \right) \nu_2 dh \\
\frac{Pr_{21,t}}{P_{11,t}} &= \left[(1+\tau_{21}) \frac{q_t}{q_{2,t}} - mc_{1,t} \Delta_{21,t}^{T} \right] \left(c_{21,t} + x_{21,t} \right) \nu_2
\end{aligned}$$

and the non-traded goods

$$\frac{Pr_{1,t}^{N}}{P_{11,t}} = q_{1,t}^{N} \left[\left(1+\tau_{1,t}^{N}\right) - mc_{1,t}^{N} \Delta_{1,t}^{N} \right] \left(c_{1,t}^{N} + \eta \left(c_{11,t} + c_{12,t} \right) \right)$$

Consolidate household budget constraint

$$
\begin{aligned}
& \Phi_{1,t}\left(\left(1-\tau_1^W\right) w_{1,t} l_{1,t}+\left(1-\tau_1^R\right) r_{1,t} k_{1i,t-1}\right)+\Phi_{1,t}\left(pr_{1,t}+tr_{1,t}\right) \\
= & \Phi_{1,t} w_{1,t} l_{1,t}+\Phi_{1,t} r_{1,t} k_{1i,t-1}+\Phi_{1,t}\left[1-mc_{1,t} \Delta_{11,t}^T\right]\left(c_{11,t}+x_{11,t}\right) \\
& +\Phi_{1,t}\left[\frac{q_t}{q_{2,t}}-mc_{1,t} \Delta_{21,t}^T\right]\left(c_{21,t}+x_{21,t}\right) \nu_2+\Phi_{1,t} q_{1,t}^N\left[1-mc_{1,t}^N \Delta_{1,t}^N\right]\left(c_{1,t}^N+\eta\left(c_{11,t}+c_{12,t}\right)\right) \\
= & +\Phi_{1,t}\left(c_{11,t}+x_{11,t}\right)+\Phi_{1,t} \frac{q_t}{q_{2,t}}\left(c_{21,t}+x_{21,t}\right) \nu_2 \\
& +\Phi_{1,t} q_{1,t}^N\left(c_{1,t}^N+\eta\left(c_{11,t}+c_{12,t}\right)\right)
\end{aligned}
$$

44. Price of government debt

$$
Q_{i,t+j}^G = \beta_i \frac{\lambda_{t+1+j}^C}{\lambda_{t+j}^C}=\beta_i \frac{\lambda_{t+1+j}^C P_{t+1+j}^C}{\lambda_{t+j}^C P_{t+j}^C} \frac{1}{\pi_{t+j}^C}
$$

45. Government budget constraint

$$
\begin{aligned}
tr_{i,t+j} = & \tau_i^r r_{i,t+j} k_{i,t-1+j}+\tau_i^w w_{i,t+j} l_{i,t+j} \\
& -\tau_{11}\left(c_{11,t}+x_{11,t}\right) \\
& -\tau_{21} \frac{q_t}{q_{2,t}}\left(c_{21,t}+x_{21,t}\right) \nu_2 \\
& -q_{1,t}^N \tau_1^N\left(c_{1,t}^N+\eta\left(c_{11,t}+c_{12,t}\right)\right)
\end{aligned}
$$

46. Nominal interest rate

$$
\frac{1}{Rs_{i,t+j}}=Q_{i,t+j}^G
$$

47. Monetary policy

$$
Rs_{i,t+j}=Rs_i\left(\frac{Rs_{i,t-1+j}}{Rs_i}\right)^{\phi_i^{rs}}\left(\frac{\pi_{i,t+j}^c}{\pi_i^c}\right)^{\beta_i \phi_i^c\left(1-\phi_i^{rs}\right)}
$$

with the steady state inflation rate π_i^c and the steady state nominal interest rate $Rs_i=\frac{\pi_i^c}{\beta_i}$.

References

Calvo, G. (1983). Staggered prices in a utility maximizing framework. *Journal of Monetary Economics 12*(3), 383–398.

Christiano, L. J., M. Eichenbaum, and C. L. Evans (1999). Monetary policy shocks: What have we learned and to what end?

Ghironi, F. (2006). Macroeconomic interdependence under incomplete markets. *Journal of International Economics 70*(2), 428–450.

Smets, F. and R. Wouters (2003). An estimated dynamic stochastic general equilibrium model of the euro area. *Journal of the European Economic Association 1*(5), 1123–1175.

Weil, P. (1989). Overlapping families of infinitely-lived agents. *Journal of Public Economics 38*(2), 183–198.

Yun, T. (1996). Nominal price rigidity, money supply endogeneity and business cycles. *Journal of Monetary Economics 37*(2), 345–370.